

Submodular Functions, Optimization, and Applications to Machine Learning

— Fall Quarter, Lecture 10 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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Nov 2nd, 2020



$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$$-f(A) + 2f(C) + f(B), \quad -f(A) + f(C) + f(B), \quad -f(A \cap B)$$



Announcements, Assignments, and Reminders

- Homework 2, due Nov 2nd, 11:59pm on our assignment dropbox (<https://canvas.uw.edu/courses/1397085/assignments>).
- Final problem on HW2 should now be first problem on HW3 (will be out soon).
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (<https://canvas.uw.edu/courses/1397085/announcements>).
- Office hours, Wed & Thur, 10:00pm at our class zoom link.

Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid
- L9(10/28): Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy, Polyhedra, Matroid Polytopes
- L10(11/2): Matroid Polytopes, Matroids → Polymatroids, Polymatroids
- L11(11/4):
- L12(11/9):
- L-(11/11): Veterans Day, Holiday
- L13(11/16):
- L14(11/18):
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

The greedy algorithm

- In combinatorial optimization, the greedy algorithm is often useful as a heuristic that can work quite well in practice.
- The goal is to choose a good subset of items, and the fundamental tenet of the greedy algorithm is to **choose next whatever currently looks best**, without the possibility of later recall or backtracking.
- Sometimes, this gives the optimal solution (we saw in Lecture 5 three greedy algorithms that can find the maximum weight spanning tree, namely Kruskal, Jarník/Prim/Dijkstra, and Borůvka's Algorithms).
- Greedy is good since it can be made to run very fast, e.g., $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working optimally is a defining property of a matroid, and is also a defining property of a polymatroid function.

Matroid and the greedy algorithm

- Let (E, \mathcal{I}) be an independence system, and we are given a non-negative modular weight function $w : E \rightarrow \mathbb{R}_+$.

Algorithm 1: The Matroid Greedy Algorithm

- 1 Set $X \leftarrow \emptyset$;
 - 2 **while** $\exists v \in E \setminus X$ s.t. $X \cup \{v\} \in \mathcal{I}$ **do**
 - 3 $v \in \operatorname{argmax} \{w(v) : v \in E \setminus X, X \cup \{v\} \in \mathcal{I}\}$;
 - 4 $X \leftarrow X \cup \{v\}$;
-

- Same as sorting items by decreasing weight w , and then choosing items in that order that retain independence.

Theorem 10.2.4

Let (E, \mathcal{I}) be an independence system. Then the pair (E, \mathcal{I}) is a matroid *if and only if* for each weight function $w \in \mathcal{R}_+^E$, Algorithm ~~1~~ above leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A normalized monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.

non-negative

Convex Polytope - key representation theorem

- A polytope can be defined in a number of ways, two of which include

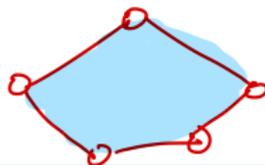
Theorem 10.2.6

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- *P is the convex hull of a finite set of points.*
- *If it is a **bounded** intersection of halfspaces, that is there exists matrix A and vector b such that*

$$P = \{x : Ax \leq b\} \tag{10.9}$$

- This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carátheodory.



Linear Programming duality forms

There are many ways to construct the dual. For example,

$$\max \{c^T x \mid x \geq 0, Ax \leq b\} = \min \{y^T b \mid y \geq 0, y^T A \geq c^T\} \quad (10.11)$$

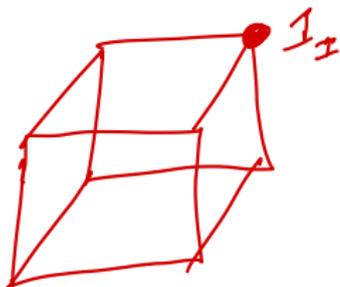
$$\max \{c^T x \mid x \geq 0, Ax = b\} = \min \{y^T b \mid y^T A \geq c^T\} \quad (10.12)$$

$$\min \{c^T x \mid x \geq 0, Ax \geq b\} = \max \{y^T b \mid y \geq 0, y^T A \leq c^T\} \quad (10.13)$$

$$\min \{c^T x \mid Ax \geq b\} = \max \{y^T b \mid y \geq 0, y^T A = c^T\} \quad (10.14)$$

Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I \in \{0, 1\}^E \subset [0, 1]^E \subset \mathbb{R}_+^E$.



Independence Polyhedra

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- Taking the convex hull, we get the **independent set polytope**, that is

$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \subseteq [0, 1]^E \quad (10.1)$$

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- Now take the rank function r of M , and define the following polyhedron:

$$P_r^+ \triangleq \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \quad (10.2)$$

Examples of P_r^+ are forthcoming.

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- Now, take any $x \in P_{\text{ind. set}}$, then we will show that that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this after a few examples of P_r^+ .

Matroid Polyhedron in 2D

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \quad (10.3)$$

- Consider this in two dimensions. We have equations of the form:

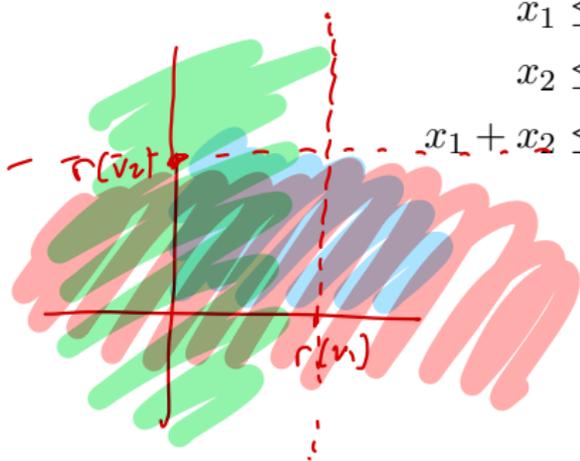
$$|E|=2$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \quad (10.4)$$

$$x_1 \leq r(\{v_1\}) \in \{0, 1\} \quad (10.5)$$

$$x_2 \leq r(\{v_2\}) \in \{0, 1\} \quad (10.6)$$

$$x_1 + x_2 \leq r(\{v_1, v_2\}) \in \{0, 1, 2\} \quad (10.7)$$



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- Consider this in two dimensions. We have equations of the form:

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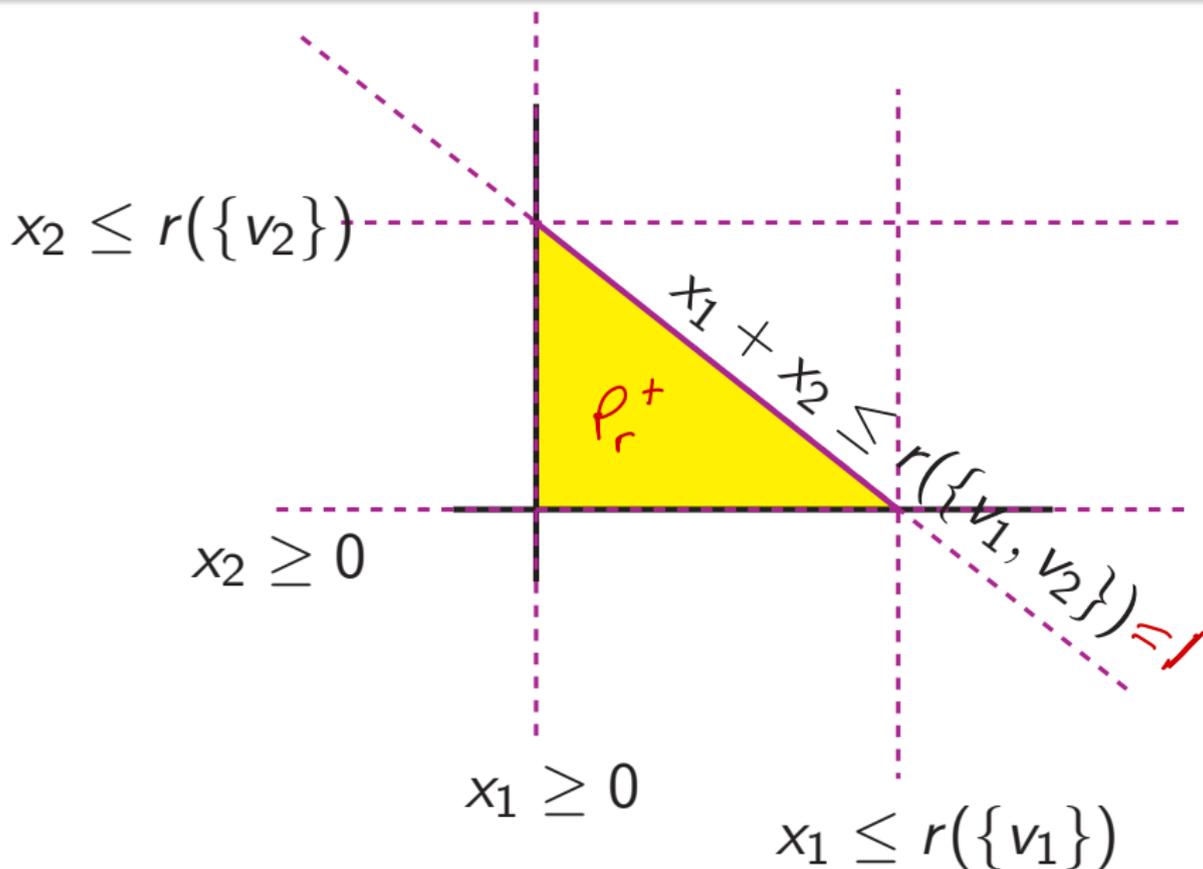
$$x_1 + x_2 \leq r(\{v_1, v_2\}) \in \{0, 1, 2\} \quad (10.7)$$

- Because r is submodular, we have

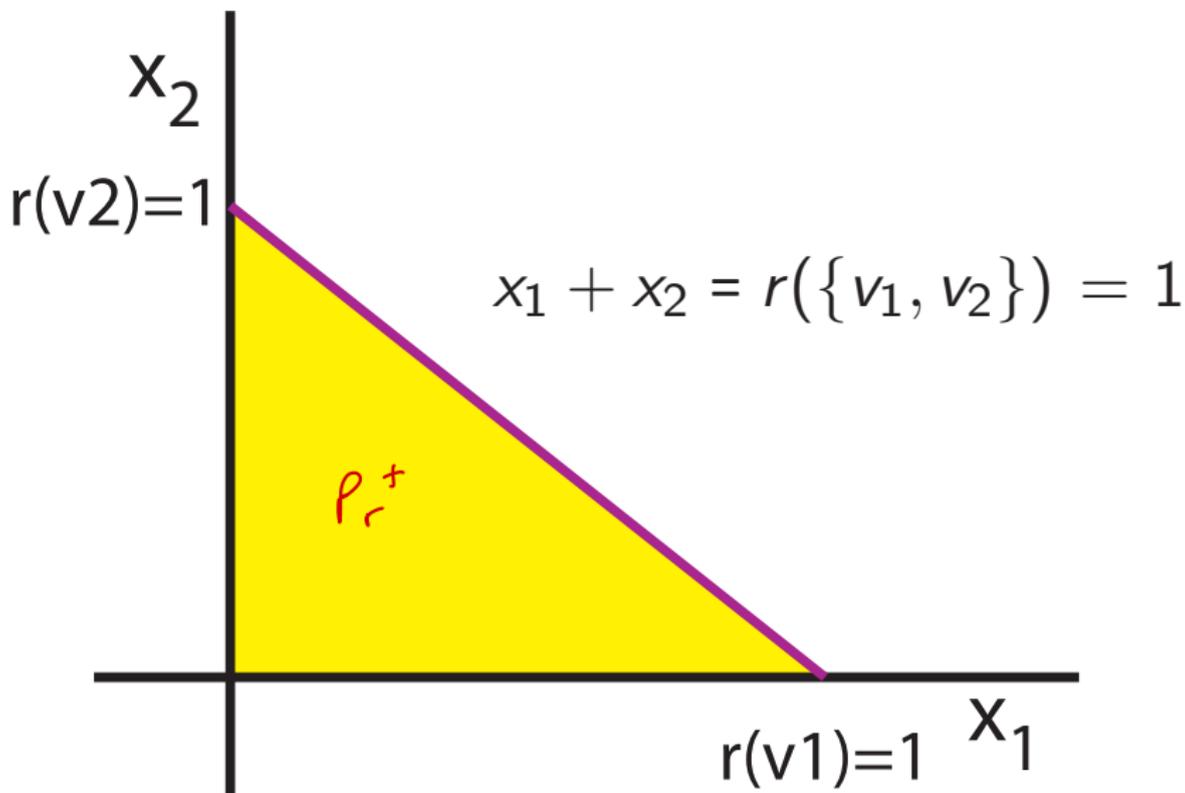
$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \quad (10.8)$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either superfluous ($r(v_1, v_2) = r(v_1) + r(v_2)$, “inactive”) or “active.”

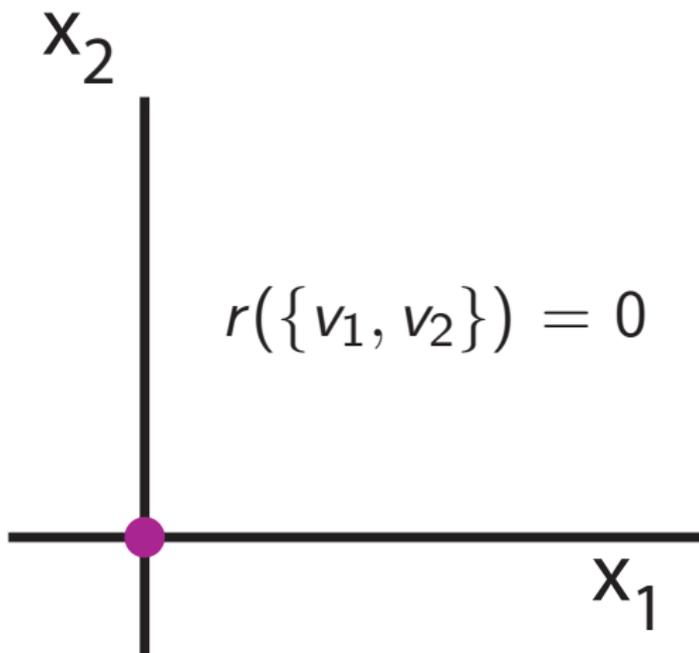
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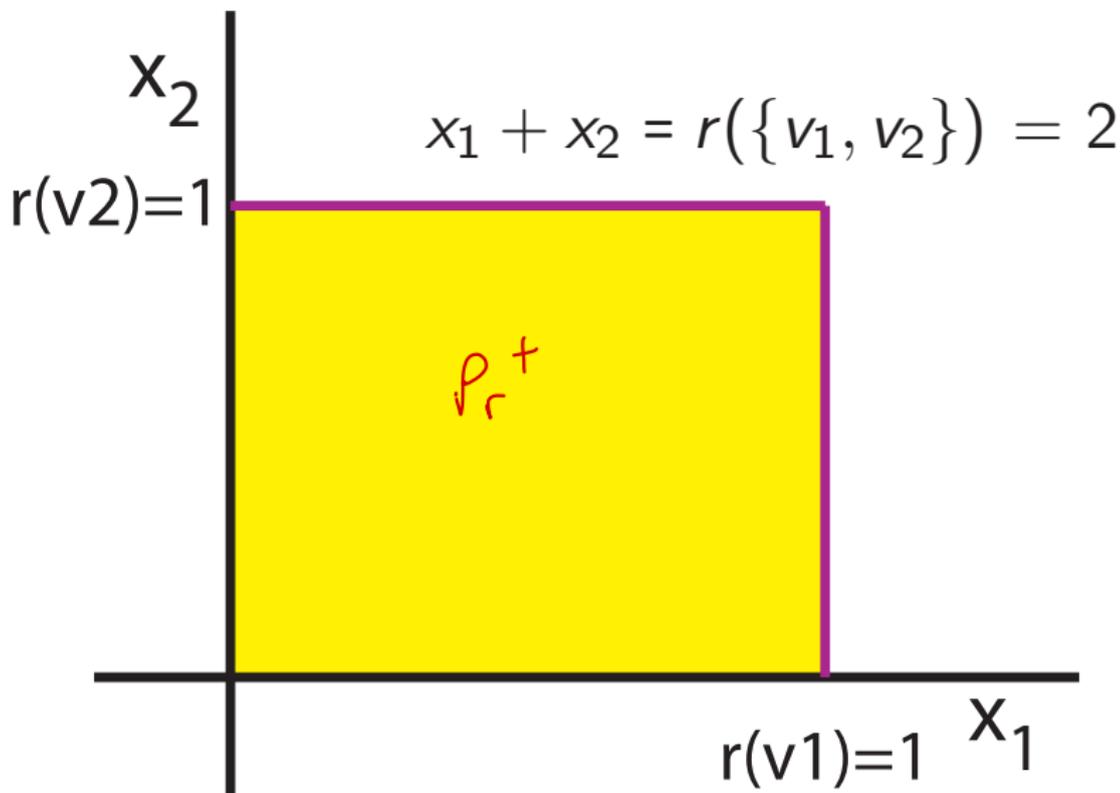
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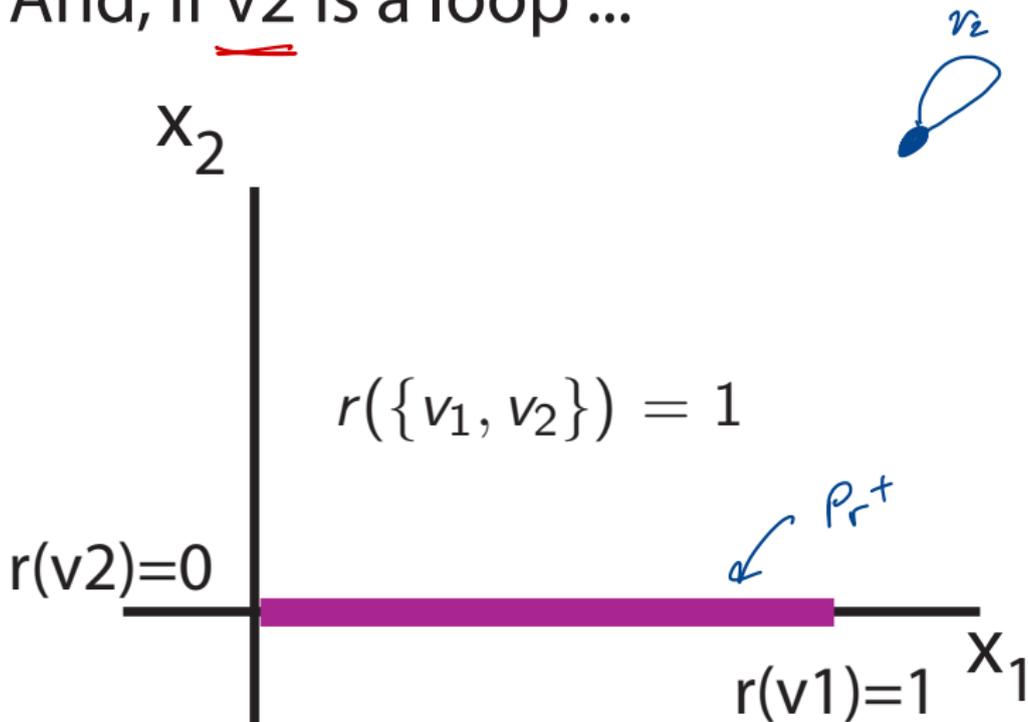


Matroid Polyhedron in 2D

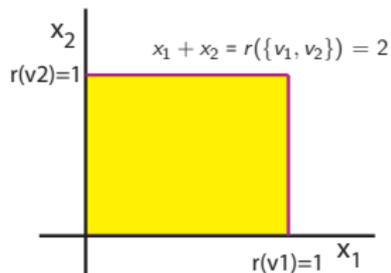
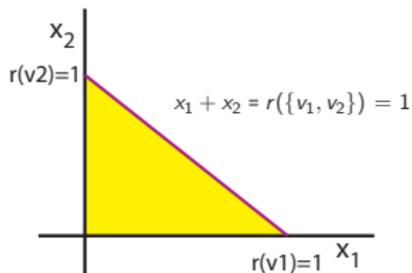
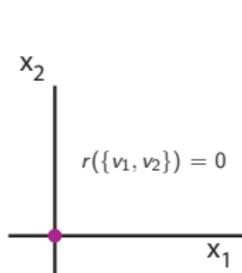


Matroid Polyhedron in 2D

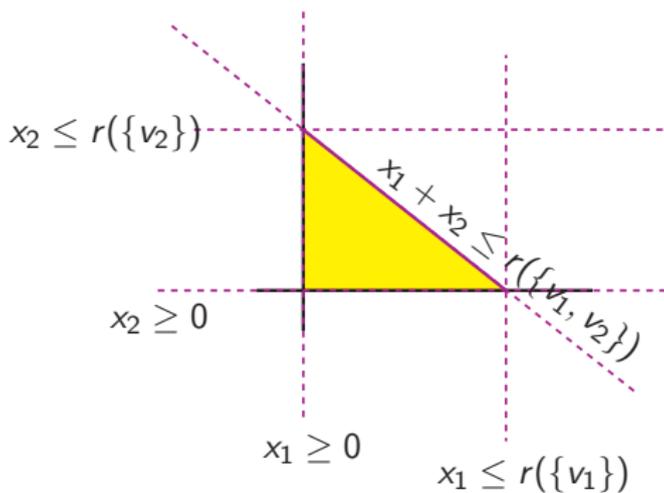
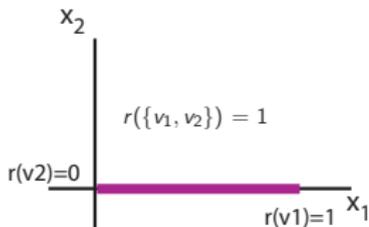
And, if v_2 is a loop ...



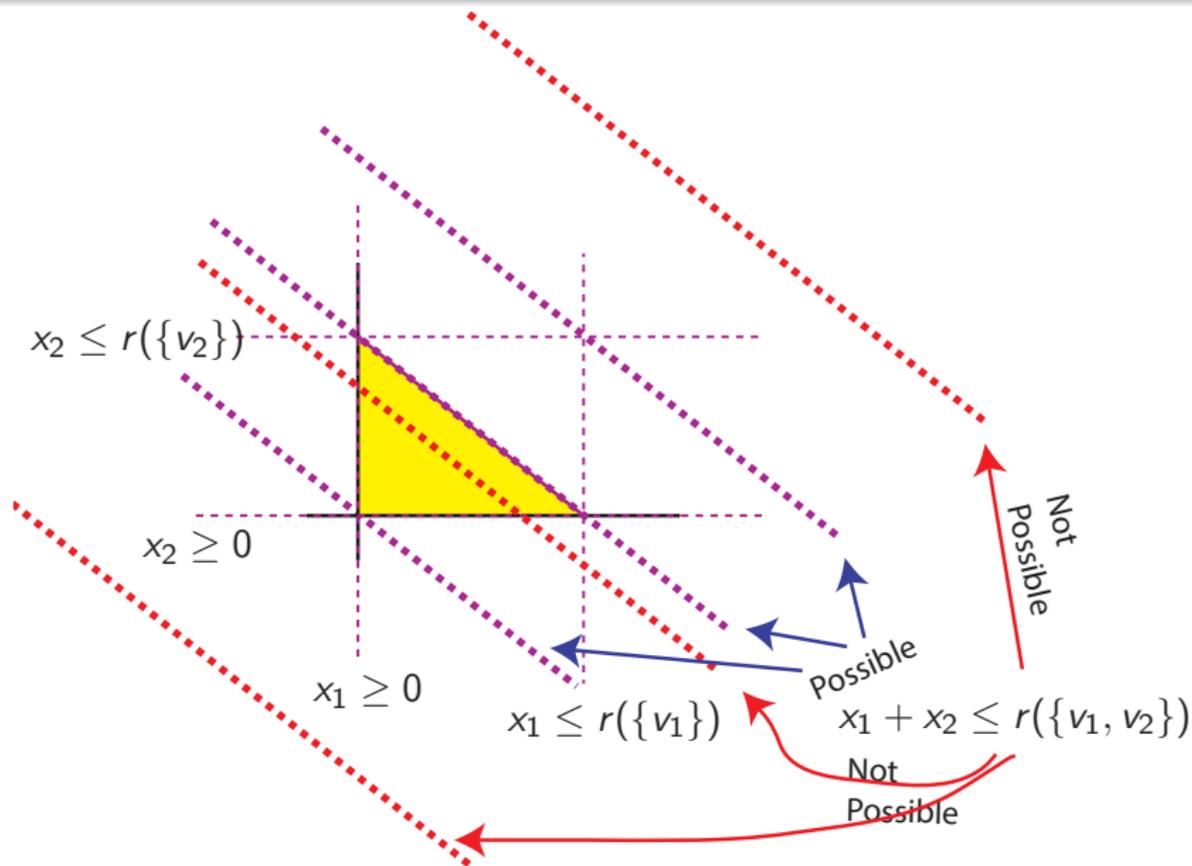
Matroid Polyhedron in 2D



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Matroid Polyhedron in 2D



Matroid Polyhedron in 3D

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \quad (10.9)$$

- Consider three dimensions, $E = \{1, 2, 3\}$. Get equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \quad (10.10)$$

$$x_1 \leq r(\{v_1\}) \quad (10.11)$$

$$x_2 \leq r(\{v_2\}) \quad (10.12)$$

$$x_3 \leq r(\{v_3\}) \quad (10.13)$$

$$x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (10.14)$$

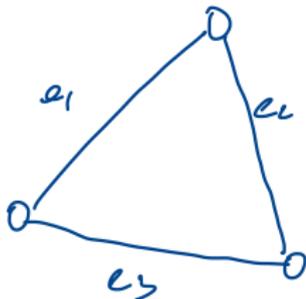
$$x_2 + x_3 \leq r(\{v_2, v_3\}) \quad (10.15)$$

$$x_1 + x_3 \leq r(\{v_1, v_3\}) \quad (10.16)$$

$$x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \quad (10.17)$$

Matroid Polyhedron in 3D

- Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.



$$E = \{e_1, e_2, e_3\}$$

- $\text{rank}(e_i) = 1$
- $\text{rank}(e_i, e_j) = 2$
 $i \neq j$
- $\text{rank}(M) = 2$

Matroid Polyhedron in 3D

- Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.
- So any set of either one or two edges is independent, and has rank equal to cardinality.

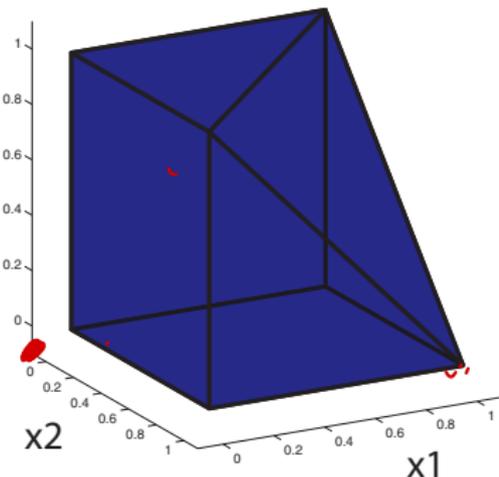
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- So any set of either one or two edges is independent, and has rank equal to cardinality.
- The set of three edges is dependent, and has rank 2.

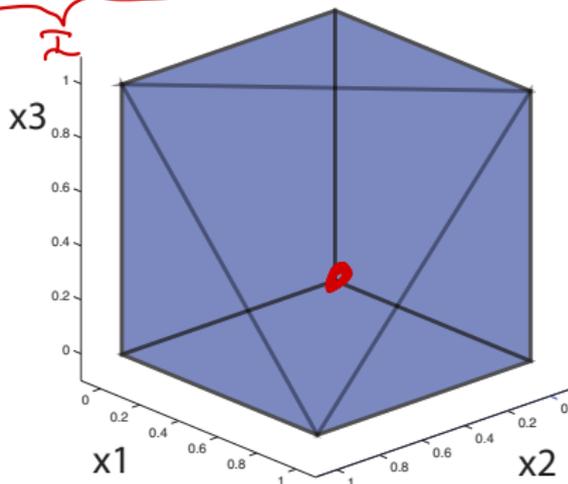
Matroid Polyhedron in 3D

Two view of P_r^+ associated with a matroid $\mathcal{M} = (E, \mathcal{I})$
 $= (\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\})$.

E
 x_3



\mathcal{I}



$$x_1 + x_2 + x_3 \leq r(e_1, e_2, e_3) = 2$$

Matroid Polyhedron in 3D

P_r^+ associated with the “free” matroid in 3D.

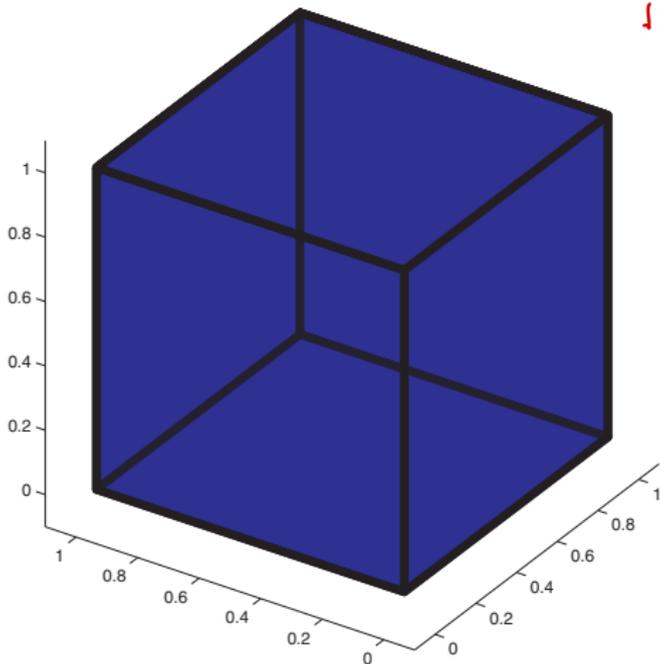
Matroid Polyhedron in 3D

P_r^+ associated with the "free" matroid in 3D.

$$M = (E, \mathcal{I})$$

$$|E| = 3$$

$$|\mathcal{I}| = 2^E$$



Review

One of

- The next two slides are from the previous lecture.
the other is from a bit earlier.

Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I \in \{0, 1\}^E \subset [0, 1]^E \subset \mathbb{R}_+^E$.
- Taking the convex hull, we get the **independent set polytope**, that is

$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \subseteq [0, 1]^E \quad (10.1)$$

- Now take the rank function r of M , and define the following polyhedron:

$$P_r^+ \triangleq \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \quad (10.2)$$

Examples of P_r^+ are forthcoming.

- Now, take any $x \in P_{\text{ind. set}}$, then we will show that that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this after a few examples of P_r^+ .

Vector, modular, incidence

- Recall, any vector $x \in \mathbb{R}^E$ can be seen as a normalized modular function, as for any $A \subseteq E$, we have

$$x(A) = \sum_{a \in A} x_a \quad (10.11)$$

- Given an $A \subseteq E$, define the incidence vector $\mathbf{1}_A \in \{0, 1\}^E$ on the unit hypercube as follows:

$$\mathbf{1}_A \stackrel{\text{def}}{=} \left\{ x \in \{0, 1\}^E : x_i = 1 \text{ iff } i \in A \right\} \quad (10.12)$$

equivalently,

$$\mathbf{1}_A(E) = \sum_{j \in E} \mathbf{1}_A(j) = |A|$$

$$\mathbf{1}_A(j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j \in A \\ 0 & \text{if } j \notin A \end{cases} \quad (10.13)$$

$$P_{\text{ind. set}} \subseteq P_r^+$$

Lemma 10.3.1 ($P_{\text{ind. set}} \subseteq P_r^+$)

- If $x \in P_{\text{ind. set}}$, then

$$x = \sum_i \lambda_i \mathbf{1}_{I_i} \quad (10.18)$$

for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

$$\sum_i \lambda_i = 1 \quad \forall i, \lambda_i \geq 0$$

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- Clearly, for such x , $x \geq 0$.
- Now, for any $A \subseteq E$,

$$x(A) = x^\top \mathbf{1}_A = \sum_i \lambda_i \mathbf{1}_{I_i}^\top \mathbf{1}_A \quad (10.19)$$

$$\mathbf{1}_I^\top \cdot \mathbf{1}_A = \mathbf{1}_{I \cap A}(E)$$

$$\sum_i \lambda_i \cdot \mathbf{1}_{I_i \cap A}(E)$$

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$$= \max_{j: I_j \subseteq A} \mathbf{1}_{I_j}(E) = \max_{I \in \mathcal{I}} |A \cap I| \quad (10.21)$$


 $|I_j|$

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$$= r(A) \quad (10.22)$$

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$$= r(A) \quad (10.22)$$

- Thus, $x \in P_r^+$ and hence $P_{\text{ind. set}} \subseteq P_r^+$.

Containment: Matroid Independence Polyhedron

- Thus, we have that:

$$\begin{aligned} P_{\text{ind. set}} &= \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \\ &\subseteq P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \end{aligned} \quad (10.23)$$

Containment: Matroid Independence Polyhedron

- Thus, we have that:

$$\begin{aligned}
 P_{\text{ind. set}} &= \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \\
 &\subseteq P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \quad (10.23)
 \end{aligned}$$

- Therefore, since $\{\mathbf{1}_I : I \in \mathcal{I}\} \subseteq \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} = P_{\text{ind. set}} \subseteq P_r^+$, we have that

$$\max \{w(I) : I \in \mathcal{I}\} \leq \max \{w^\top x : x \in P_{\text{ind. set}}\} \quad (10.24)$$

$$\leq \max \{w^\top x : x \in P_r^+\} \quad (10.25)$$

$|E|$
 2

Containment: Matroid Independence Polyhedron

- Thus, we have that:

$$\begin{aligned}
 P_{\text{ind. set}} &= \text{conv} \{ \cup_{I \in \mathcal{I}} \{ \mathbf{1}_I \} \} \\
 &\subseteq P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (10.23)
 \end{aligned}$$

- Therefore, since $\{ \mathbf{1}_I : I \in \mathcal{I} \} \subseteq \text{conv} \{ \cup_{I \in \mathcal{I}} \{ \mathbf{1}_I \} \} = P_{\text{ind. set}} \subseteq P_r^+$, we have that

$$\max \{ w(I) : I \in \mathcal{I} \} \leq \max \{ w^\top x : x \in P_{\text{ind. set}} \} \quad (10.24)$$

$$\leq \max \{ w^\top x : x \in P_r^+ \} \quad (10.25)$$

- In fact, the two polyhedra $P_{\text{ind. set}}$ and P_r^+ are identical (and thus both are polytopes). We'll show this in the next few theorems.

Maximum weight independent set via greedy weighted rank

Theorem 10.3.2

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r , then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \dots \subset U_n \subseteq V$ such that

$$\checkmark \quad w^*(w) = \max \{w(I) \mid I \in \mathcal{I}\} = \sum_{i=1}^n \lambda_i r(U_i) \quad (10.26)$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i} \quad (10.27)$$

Maximum weight independent set via weighted rank

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$w = (w_1, w_2, \dots, w_n)^T$$

$$\begin{aligned} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} &= (w_1 - w_2) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3) \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \\ &\dots + (w_{n-1} - w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \end{aligned} \quad (10.28)$$

Maximum weight independent set via weighted rank

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$\begin{aligned}
 \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} &= \underbrace{(w_1 - w_2)}_{\geq 0} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \underbrace{(w_2 - w_3)}_{\geq 0} \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \\
 &\quad \dots + (w_{n-1} - w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{10.28}
 \end{aligned}$$

- If we can take w in non-increasing order ($w_1 \geq w_2 \geq \dots \geq w_n$), then each coefficient of the vectors is non-negative (except possibly the last one, w_n).

Maximum weight independent set via weighted rank

Proof.

- Again assuming $w \in \mathbb{R}_+^V$, w.l.o.g. order elements of V non-increasing by w so (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$

Maximum weight independent set via weighted rank

Proof.

- Again assuming $w \in \mathbb{R}_+^E$, w.l.o.g. order elements of V non-increasing by w so (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$
- Define the sets U_i based on this order as follows, for $i = 0, \dots, n$

$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \dots, v_i\} \quad (10.29)$$

Note that $U_0 = \emptyset$ and

$$\mathbf{1}_{U_0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{1}_{U_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{1}_{U_\ell} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ etc.}$$

$\left. \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \right\} \ell \times$
 $\left. \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right\} (n - \ell) \times$

Maximum weight independent set via weighted rank

Proof.

- Again assuming $w \in \mathbb{R}_+^E$, w.l.o.g. order elements of V non-increasing by w so (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$
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$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \dots, v_i\} \quad (10.29)$$

- Define the set I as those elements where the rank increases, i.e.:

$$I \stackrel{\text{def}}{=} \{v_i \mid r(U_i) > r(U_{i-1})\}. \quad (10.30)$$

Hence, given an i with $v_i \notin I$, $r(U_i) = r(U_{i-1})$.

Maximum weight independent set via weighted rank

Proof.

- Again assuming $w \in \mathbb{R}_+^E$, w.l.o.g. order elements of V non-increasing by w so (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$
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Hence, given an i with $v_i \notin I$, $r(U_i) = r(U_{i-1})$.

- Therefore, I is the output of the greedy algorithm for $\max \{w(I) \mid I \in \mathcal{I}\}$. *since items v_i are ordered decreasing by $w(v_i)$, and we only choose the ones that increase the rank, which means they don't violate independence.*

Maximum weight independent set via weighted rank

Proof.

- Again assuming $w \in \mathbb{R}_+^E$, w.l.o.g. order elements of V non-increasing by w so (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$
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- Therefore, I is the output of the greedy algorithm for $\max \{w(I) \mid I \in \mathcal{I}\}$.
- And therefore, I is a maximum weight independent set (can even be a base, actually).

Maximum weight independent set via weighted rank

Proof.

- Now, we define λ_i as follows

$$0 \leq \lambda_i \stackrel{\text{def}}{=} w(v_i) - w(v_{i+1}) \text{ for } i = 1, \dots, n-1 \quad (10.31)$$

$$\lambda_n \stackrel{\text{def}}{=} w(v_n) \quad (10.32)$$

Maximum weight independent set via weighted rank

Proof.

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$$\lambda_n \stackrel{\text{def}}{=} w(v_n) \quad (10.32)$$

- And the weight of the independent set $w(I)$ is given by

$$w(I) = \sum_{v \in I} w(v) = \quad (10.34)$$

Maximum weight independent set via weighted rank

Proof.

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$$w(I) = \sum_{v \in I} w(v) = \sum_{i=1}^n w(v_i) (r(U_i) - r(U_{i-1})) \quad (10.33)$$

$\in \{0, 1\}$

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Maximum weight independent set via weighted rank

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$$= w(v_n)r(U_n) + \sum_{i=1}^{n-1} (w(v_i) - w(v_{i+1}))r(U_i) \quad (10.34)$$

Since $r(\emptyset) = 0$

Maximum weight independent set via weighted rank

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$$w(I) = \sum_{v \in I} w(v) = \sum_{i=1}^n w(v_i) (r(U_i) - r(U_{i-1})) \quad (10.33)$$

$$= w(v_n) r(U_n) + \sum_{i=1}^{n-1} (w(v_i) - w(v_{i+1})) r(U_i) = \sum_{i=1}^n \lambda_i r(U_i) \quad (10.34)$$

Maximum weight independent set via weighted rank

Proof.

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- Since we ordered v_1, v_2, \dots non-increasing by w , for all i , and since $w \in \mathbb{R}_+^E$, we have $\lambda_i \geq 0$



Linear Program LP

Consider the linear programming primal problem

$$\begin{aligned} & \text{maximize} && w^\top x \\ & \text{subject to} && x_v \geq 0 && (v \in V) \\ & && x(U) \leq r(U) && (\forall U \subseteq V) \end{aligned} \tag{10.35}$$

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And its convex dual (note $y \in \mathbb{R}_+^{2^n}$, y_U is a scalar element within this exponentially big vector):

$$\begin{aligned}
 & \text{minimize} && \sum_{U \subseteq V} y_U r(U), \\
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 & && \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w
 \end{aligned} \tag{10.36}$$

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 \end{aligned} \tag{10.36}$$

Thanks to strong duality, the solutions to these are equal to each other.

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 \end{aligned} \tag{10.37}$$

- This is identical to the problem

$$\max w^\top x \text{ such that } x \in P_r^+ \tag{10.38}$$

where, again, $P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\}$.

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where, again, $P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\}$.

- Therefore, since $P_{\text{ind. set}} \subseteq P_r^+$, the above problem can only have a larger solution. I.e.,

$$\max w^\top x \text{ s.t. } x \in P_{\text{ind. set}} \leq \max w^\top x \text{ s.t. } x \in P_r^+. \tag{10.39}$$

Polytope equivalence

- Hence, we have the following relations:

$$\max \{w(I) : I \in \mathcal{I}\} \leq \max \{w^\top x : x \in P_{\text{ind. set}}\} \quad (10.40)$$

$$\leq \max \{w^\top x : x \in P_r^+\} \quad (10.41)$$

$$\stackrel{\text{def}}{=} \alpha_{\min} = \min \left\{ \sum_{U \subseteq V} y_U r(U) : \forall U, y_U \geq 0; \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w \right\} \quad (10.42)$$

$$\leq \sum_{U \subseteq V} \tilde{y}_U r(U) \quad \forall \tilde{y}$$

$$\tilde{y}_U \geq 0$$

s.t.

$$\sum_{U \subseteq V} \tilde{y}_U \mathbf{1}_U \geq w$$

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- Theorem 10.3.2 states that

$$\max \{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^n \lambda_i r(U_i) \quad (10.43)$$

for the chain of U_i 's and $\lambda_i \geq 0$ that satisfies $w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i}$ (i.e., the r.h.s. of Eq. 10.43 is feasible w.r.t. the dual LP).

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- Therefore, we ~~also~~ ^{what we} have $\max \{w(I) : I \in \mathcal{I}\} \leq \alpha_{\min}$ and ~~we also have~~ ^{that}

$$\max \{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^n \lambda_i r(U_i) \geq \alpha_{\min} \quad (10.44)$$

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- Hence, we have the following relations:

$$\max \{w(I) : I \in \mathcal{I}\} \leq \max \{w^\top x : x \in P_{\text{ind. set}}\} \quad (10.40)$$

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- Therefore, all the inequalities above are equalities.

Polytope equivalence

- Hence, we have the following relations:

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- Therefore, all the inequalities above are equalities.
- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r^+ = P_{\text{ind. set}}$

Polytope equivalence

- Hence, we have the following relations:

$$\max \{w(I) : I \in \mathcal{I}\} = \max \{w^\top x : x \in P_{\text{ind. set}}\} \quad (10.40)$$

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- Therefore, all the inequalities above are equalities.
- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r^+ = P_{\text{ind. set}}$
- That is, we have just proven:

Theorem 10.3.3

$$P_r^+ = P_{\text{ind. set}} \quad (10.45)$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.

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- Taking the convex hull, we get the **independent set polytope**, that is

$$P_{\text{ind. set}} = \text{conv} \{ \cup_{I \in \mathcal{I}} \{ \mathbf{1}_I \} \} \quad (10.46)$$

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- Now take the rank function r of M , and define the following polytope:

$$P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (10.47)$$

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Theorem 10.3.4

$$P_r^+ = P_{\text{ind. set}} \quad (10.48)$$

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq 10.40, the LP problem with exponential number of constraints $\max \{w^\top x : x \in P_r^+\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

Greedy solves a linear programming problem

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Theorem 10.3.5

The LP problem $\max \{w^\top x : x \in P_r^+\}$ can be solved exactly using the greedy algorithm.

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The LP problem $\max \{w^\top x : x \in P_r^+\}$ can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since P_r^+ is described as the intersection of an exponential number of half spaces).

Greedy solves a linear programming problem

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The LP problem $\max \{w^\top x : x \in P_r^+\}$ can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since P_r^+ is described as the intersection of an exponential number of half spaces).

- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

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$$x(A) \leq r(A) \quad \forall A \subseteq V \tag{10.50}$$

$$x(V) = r(V) \tag{10.51}$$

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- By essentially the same argument as above (**Exercise:**), we can shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 10.49- 10.51 above.

Base Polytope Equivalence

- Consider convex hull of indicator vectors just of the **bases** of a matroid, rather than all of the independent sets.
- Consider a polytope defined by the following constraints:

$$x \geq 0 \tag{10.49}$$

$$x(A) \leq r(A) \quad \forall A \subseteq V \tag{10.50}$$

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- Note the third requirement, $x(V) = r(V)$.
- By essentially the same argument as above (**Exercise:**), we can shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 10.49- 10.51 above.
- What does this look like? The base polytope.

Spanning set polytope

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Theorem 10.3.6

The spanning set polytope is determined by the following equations:

$$0 \leq x_e \leq 1 \quad \text{for } e \in E \quad (10.52)$$

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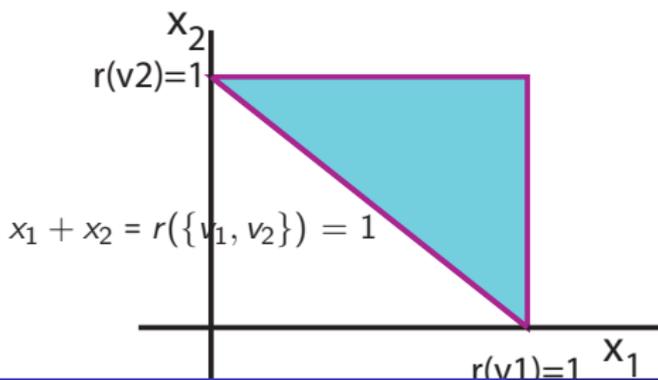
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- Example of spanning set polytope in 2D.



Spanning set polytope

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- Recall that any A is spanning in M iff $E \setminus A$ is independent in M^* (the dual matroid).

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Spanning set polytope

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- Recall that any A is spanning in M iff $E \setminus A$ is independent in M^* (the dual matroid).
- For any $x \in \mathbb{R}^E$, we have that

$$x \in P_{\text{spanning}}(M) \Leftrightarrow 1 - x \in P_{\text{ind. set}}(M^*) \quad (10.54)$$

as we show next ...

...

Spanning set polytope

... proof continued.

- This follows since if $x \in P_{\text{spanning}}(M)$, we can represent x as a convex combination:

$$x = \sum_i \lambda_i \mathbf{1}_{A_i} \quad (10.55)$$

where A_i is spanning in M .

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- Consider

$$\mathbf{1} - x = \mathbf{1}_E - x = \mathbf{1}_E - \sum_i \lambda_i \mathbf{1}_{A_i} = \sum_i \lambda_i \mathbf{1}_{E \setminus A_i}, \quad (10.56)$$

which follows since $\sum_i \lambda_i \mathbf{1} = \mathbf{1}_E$, so $\mathbf{1} - x$ is a convex combination of independent sets in M^* and so $\mathbf{1} - x \in P_{\text{ind. set}}(M^*)$.

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Spanning set polytope

... proof continued.

- which means, from the definition of $P_{\text{ind. set}}(M^*)$, that

$$\mathbf{1} - x \geq 0 \quad (10.57)$$

$$\mathbf{1}_A - x(A) = |A| - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E \quad (10.58)$$

And we know the dual rank function is

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- By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it is worth showing a result for a general submodular function first.
- Henceforth, we will skip between submodular functions and matroids, each lecture talking less about matroids specifically and taking more about submodular functions more generally ...

Maximal points in a set

- Regarding sets, a subset X of S is a **maximal** subset of S possessing a given property \mathfrak{P} if X possesses property \mathfrak{P} and no set properly containing X (i.e., any $X' \supset X$ with $X' \setminus X \subseteq \cancel{V \setminus X}$) possesses \mathfrak{P} .

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- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector x is **maximal within P** if it is the case that for any $\epsilon > 0$, and for all directions $e \in E$, we have that

$$x + \epsilon \mathbf{1}_e \notin P \quad (10.61)$$

Maximal points in a set

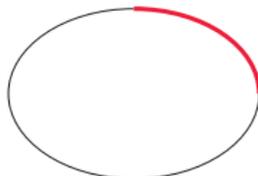
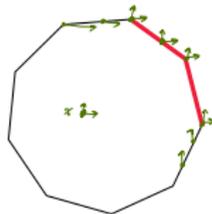
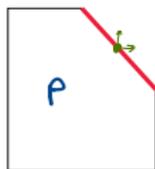
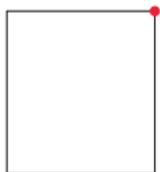
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Basis

$$\mathbf{1}_e, e \in E \quad (10.61)$$

- Examples of maximal regions (in red)



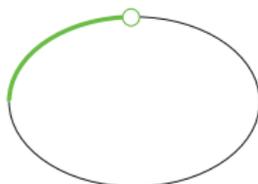
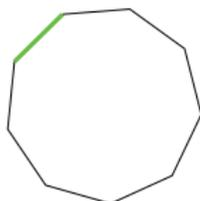
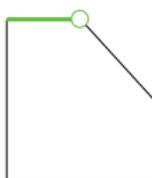
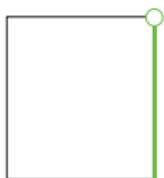
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- Examples of non-maximal regions (in green)



Review from Lecture 6

- The next slide comes from Lecture 6.

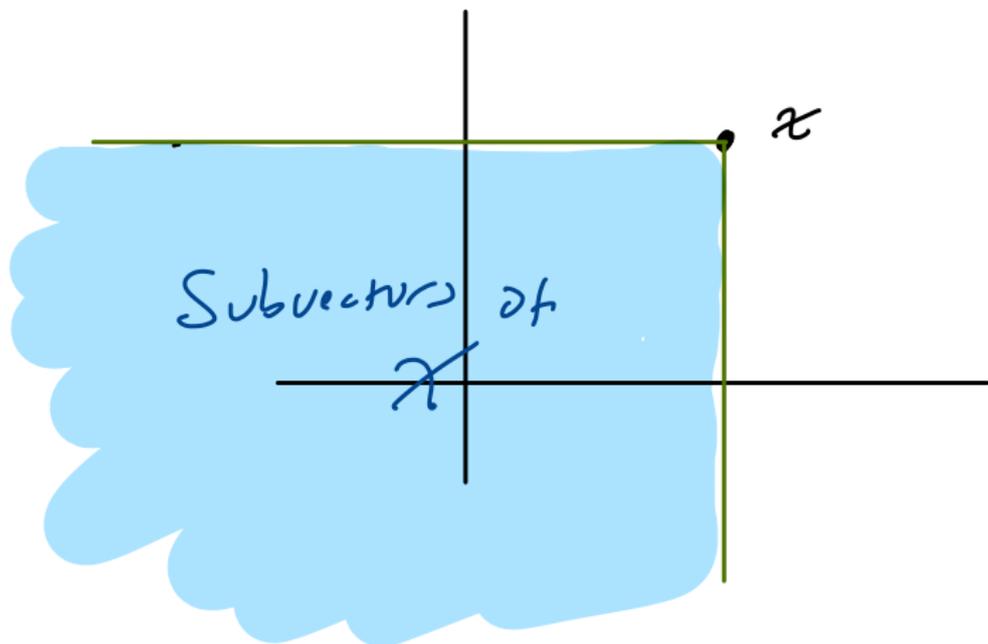
Matroids, independent sets, and bases

- Independent sets: Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called **independent** if $A \in \mathcal{I}$ and otherwise A is called **dependent**.
- A base of $U \subseteq E$: For $U \subseteq E$, a subset $B \subseteq U$ is called a **base** of U if B is inclusionwise maximally independent subset of U . That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.
- A **base of a matroid**: If $U = E$, then a “base of E ” is just called a **base** of the matroid M (this corresponds to a **basis** in a linear space, or a **spanning forest** in a graph, or a **spanning tree** in a connected graph).

P -basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 10.4.1 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).



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Here, by y being “maximal”, we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P , and a subvector of x).

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In still other words: y is a P -basis of x if:

- 1 $y \leq x$ (y is a subvector of x); and
- 2 $y \in P$ and $y + \epsilon \mathbf{1}_e \notin P$ for all $e \in E$ where $y(e) < x(e)$ and $\forall \epsilon > 0$ (y is maximal P -contained).

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max \{|I| : I \subseteq A, I \in \mathcal{I}\} = \max_{I \in \mathcal{I}} |A \cap I| \quad (10.62)$$

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- In general, might be hard to compute and/or have ill-defined properties. Next, we look at an object that restrains and cultivates this form of rank.

Polymatroidal polyhedron (or a “polymatroid”)

Definition 10.4.3 (polymatroid)

A **polymatroid** is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

- 1 $0 \in P$
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 - Condition 3 restated (again): For every vector $x \in \mathbb{R}_+^E$, every maximal independent (i.e., $\in P$) subvector y of x has the same component sum $y(E) = \text{rank}(x)$.

Polymatroidal polyhedron (or a “polymatroid”)

Definition 10.4.3 (polymatroid)

A **polymatroid** is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

- 1 $0 \in P$
- 2 If $y \leq x \in P$ then $y \in P$ (called **down monotone**).
- 3 For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P -basis of x), has the same component sum $y(E)$
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 - Condition 3 restated (yet again): All P -bases of x have the same component sum.

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- Vectors within P (i.e., any $y \in P$) are called **independent**, and any vector outside of P is called **dependent**.
 - Since all P -bases of x have the same component sum, if \mathcal{B}_x is the set of P -bases of x , then $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Matroid and Polymatroid: side-by-side

A Matroid is:

A Polymatroid is:

Matroid and Polymatroid: side-by-side

A Matroid is:

- 1 a set system (E, \mathcal{I})

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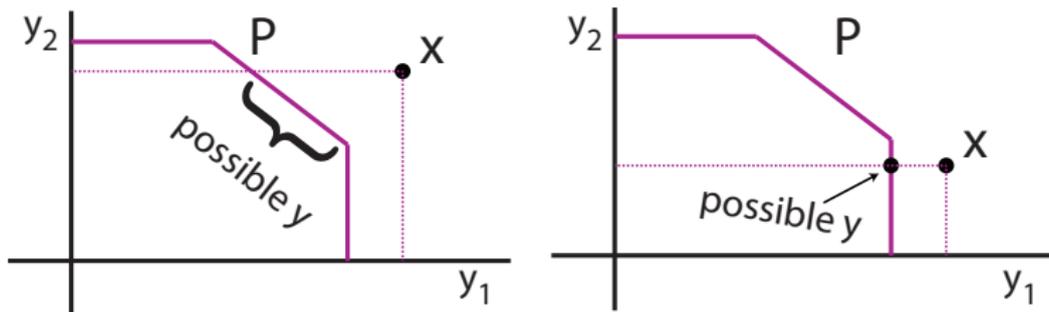
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A Polymatroid is:

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- 3 down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$
- 4 any maximal vector y in P , bounded by another vector x , has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).

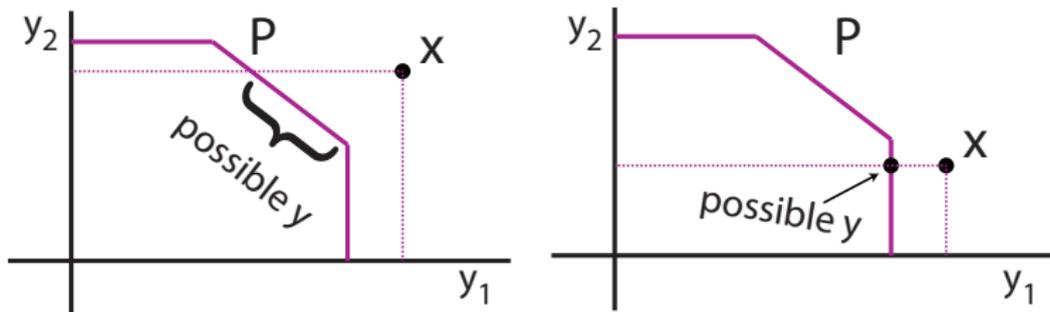
Polymatroidal polyhedron (or a “polymatroid”)



Left: \exists multiple maximal $y \leq x$ Right: \exists only one maximal $y \leq x$,

- Polymatroid condition here: \forall maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 = \text{const}$.

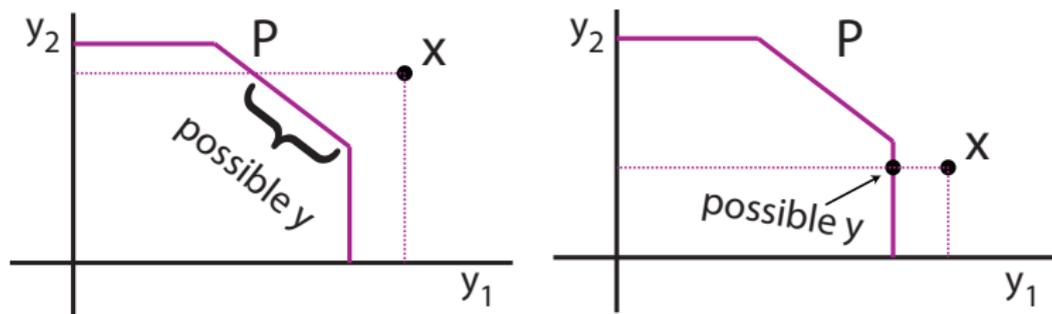
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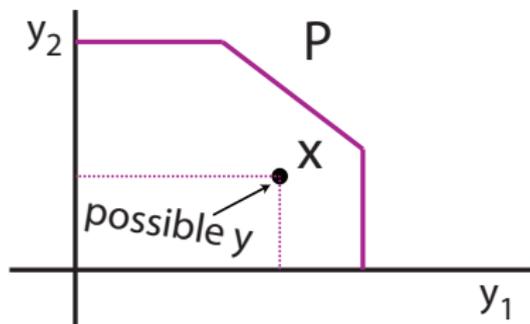
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- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such y must have the same value $y(E)$.
- On the right, there is only one maximal $y \in P$. Since there is only one, the condition on the same value of $y(E)$, $\forall y$ is vacuous.

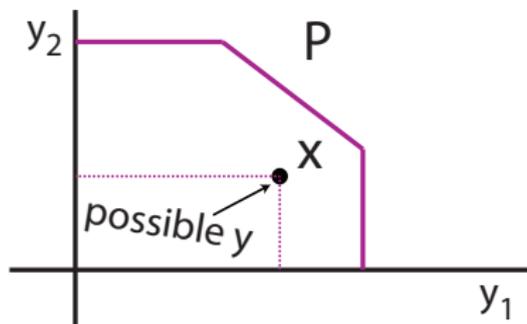
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\exists only one maximal $y \leq x$.

- If $x \in P$ already, then x is its own P -basis, i.e., it is a **self P -basis**.

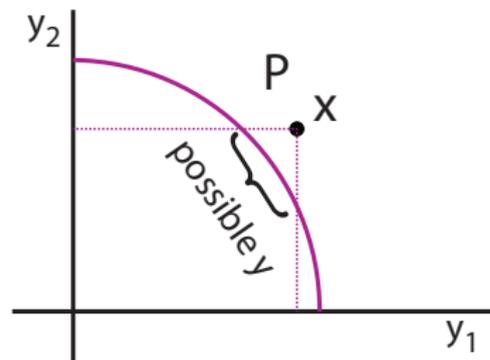
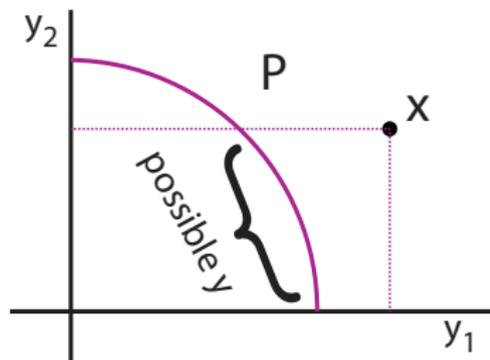
Polymatroidal polyhedron (or a “polymatroid”)



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- If $x \in P$ already, then x is its own P -basis, i.e., it is a **self P -basis**.
- In a matroid, a base of A is the maximally contained independent set. If A is already independent, then A is a self-base of A (as we saw in previous Lectures)

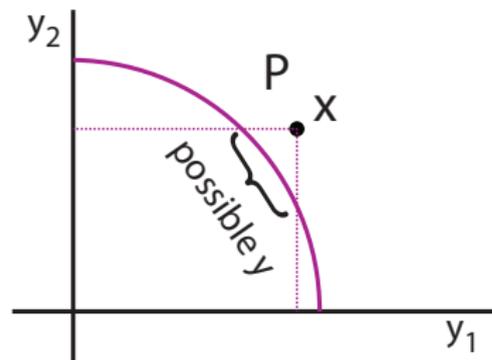
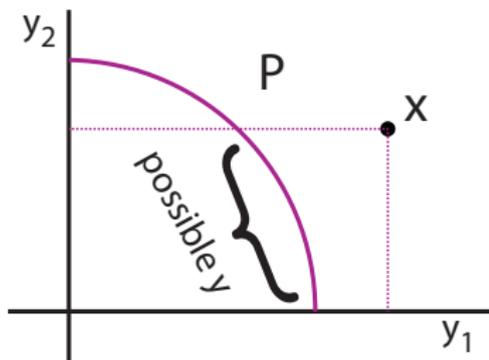
Polymatroid as well?



Left and right: \exists multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such y must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 = \text{const.} \neq y_1 + y_2$, we see this is not a polymatroid.

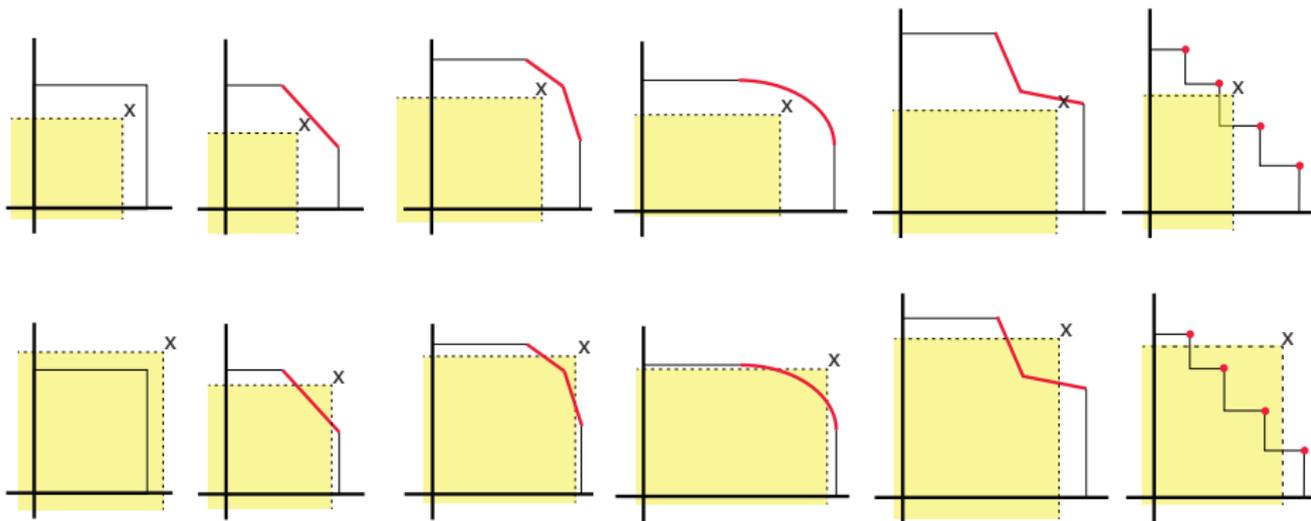
Polymatroid as well? no



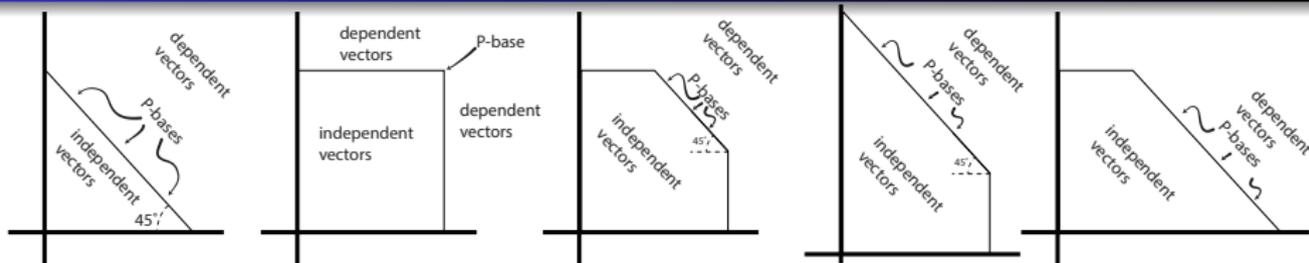
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- On the right, we have a similar situation, just the set of potential values that must have the $y(E)$ condition changes, but the values of course are still not constant.

Other examples: Polymatroid or not?



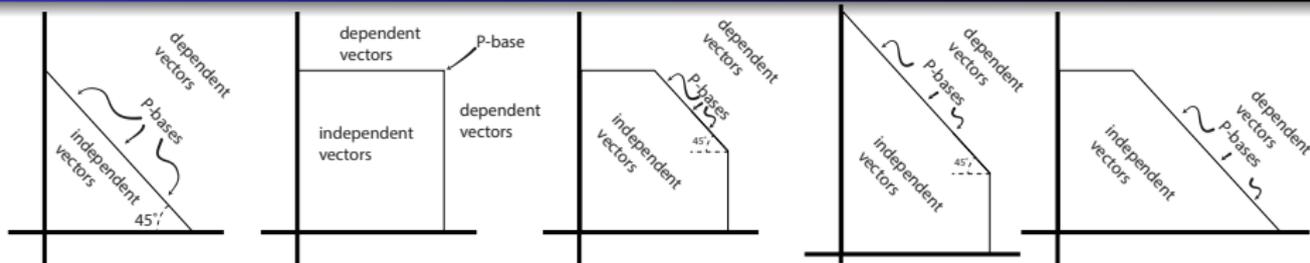
Some possible polymatroid forms in 2D



It appears that we have five possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

- 1 On the left: full dependence between v_1 and v_2

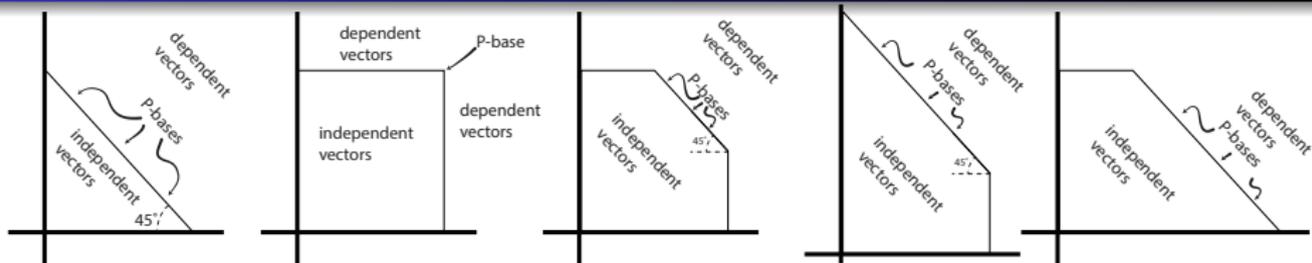
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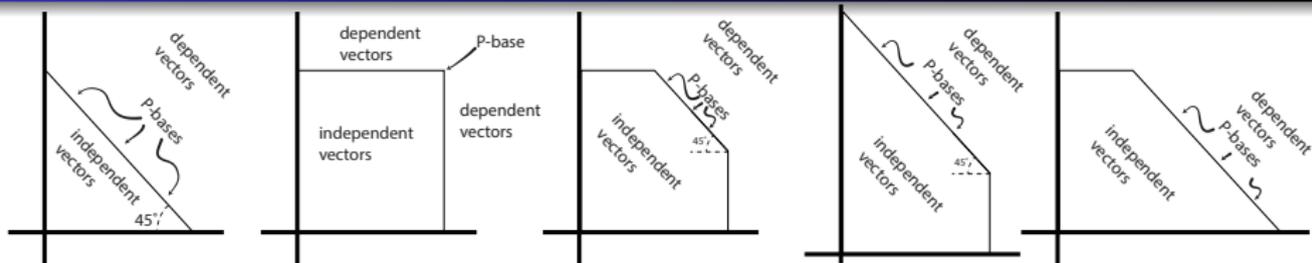
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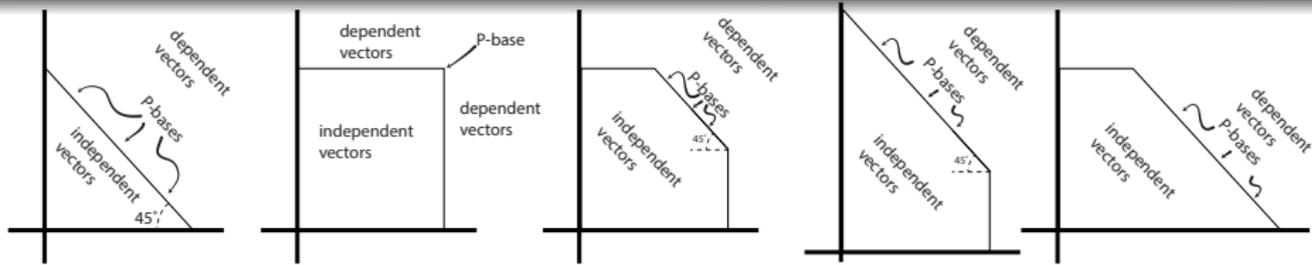
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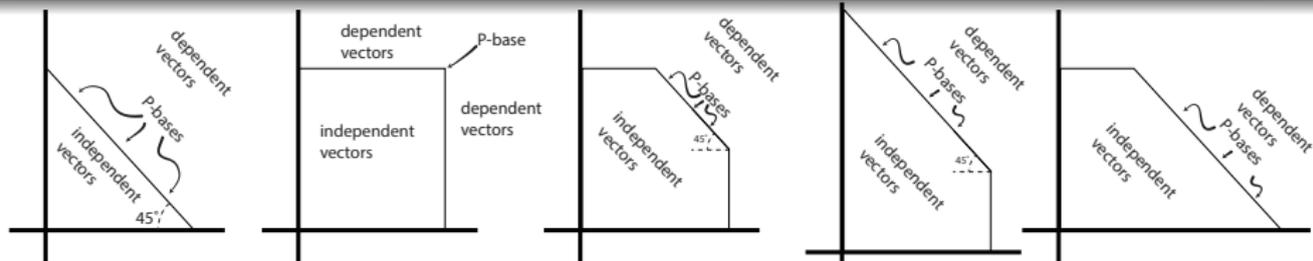
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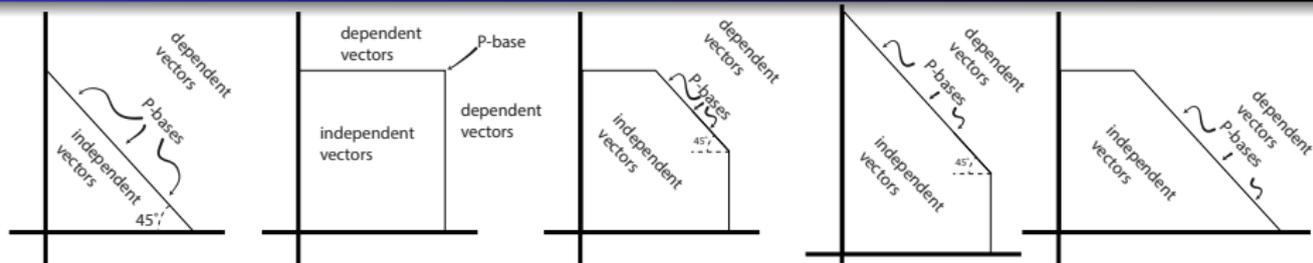
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 - Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.
 - The set of P -bases for a polytope is called the **base polytope**.

Polymatroidal polyhedron (or a “polymatroid”)

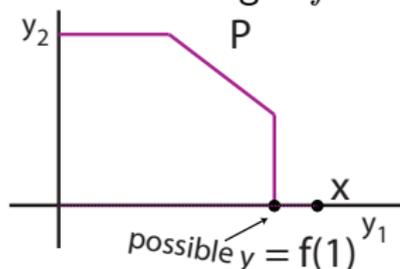
- Note that if x contains any zeros (i.e., suppose that $x \in \mathbb{R}_+^E$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so S indicates the non-zero elements, or $S = \text{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$.

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- Therefore, in this case, it is the non-zero elements of x , corresponding to elements S (i.e., the support $\text{supp}(x)$ of x), determine the common component sum.
- For the case of either $x \notin P$ or right at the boundary of P , we might give a “name” to this component sum, let's say $f(S)$ for any given set S of non-zero elements of x . We could name $\text{rank}(\frac{1}{\epsilon} \mathbf{1}_S) \triangleq f(S)$ for ϵ small enough. What kind of function might f be?



Polymatroid function and its polyhedron.

Definition 10.4.4

A **polymatroid function** is a real-valued function f defined on subsets of E which is normalized, non-decreasing, and submodular. That is we have

- ① $f(\emptyset) = 0$ (normalized)
- ② $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
- ③ $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron P_f^+ associated with a polymatroid function as follows

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \quad (10.64)$$

$$= \{y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E\} \quad (10.65)$$

Associated polyhedron with a polymatroid function

$$P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\} \quad (10.66)$$

- Consider this in three dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \quad (10.67)$$

$$x_1 \leq f(\{v_1\}) \quad (10.68)$$

$$x_2 \leq f(\{v_2\}) \quad (10.69)$$

$$x_3 \leq f(\{v_3\}) \quad (10.70)$$

$$x_1 + x_2 \leq f(\{v_1, v_2\}) \quad (10.71)$$

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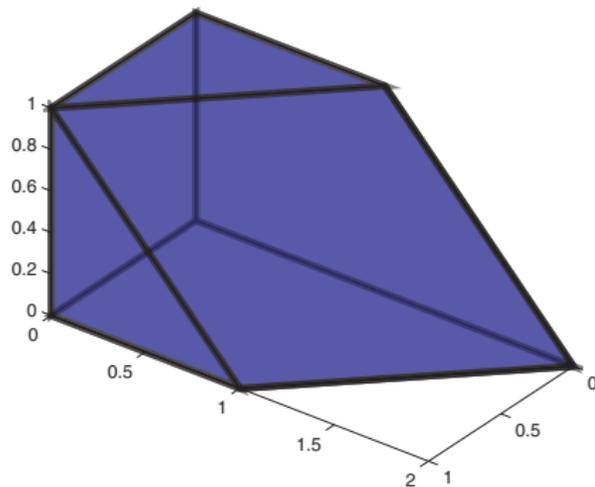
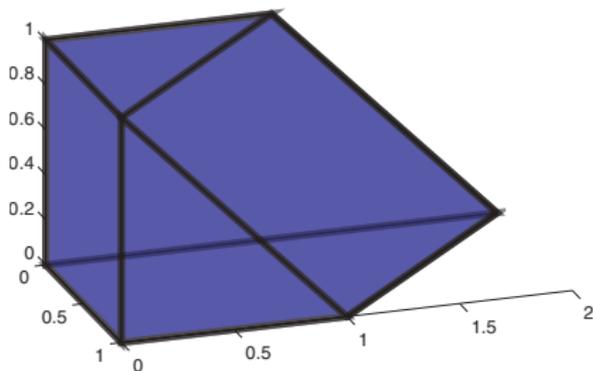
$$x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\}) \quad (10.74)$$

Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.

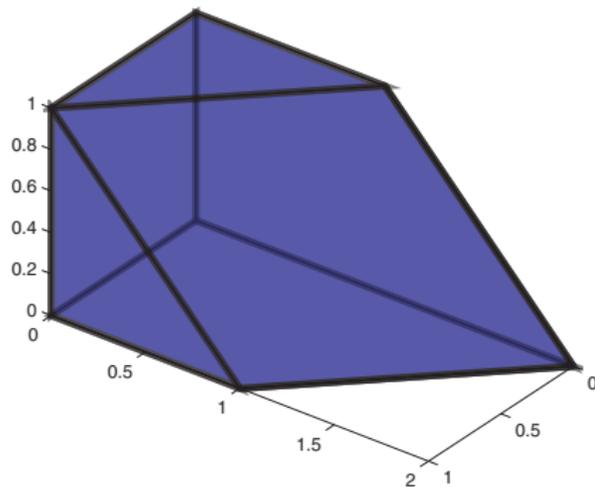
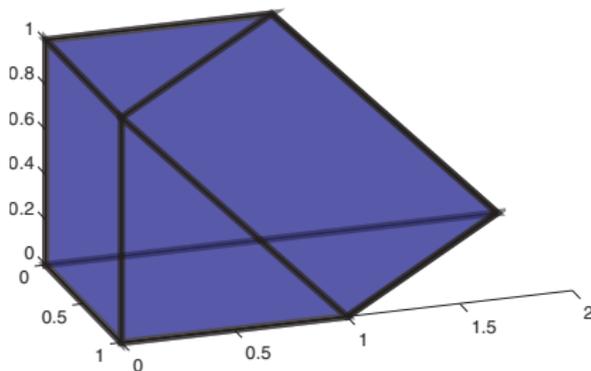
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- Observe: P_f^+ (at two views):



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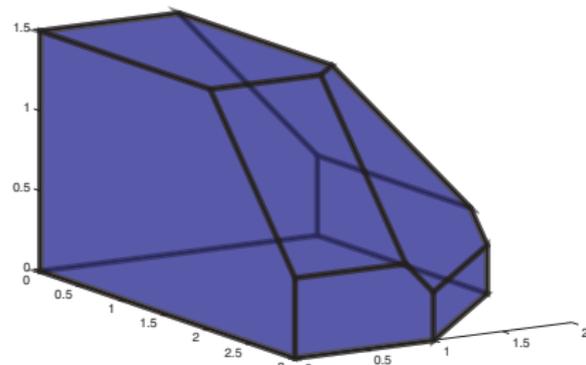
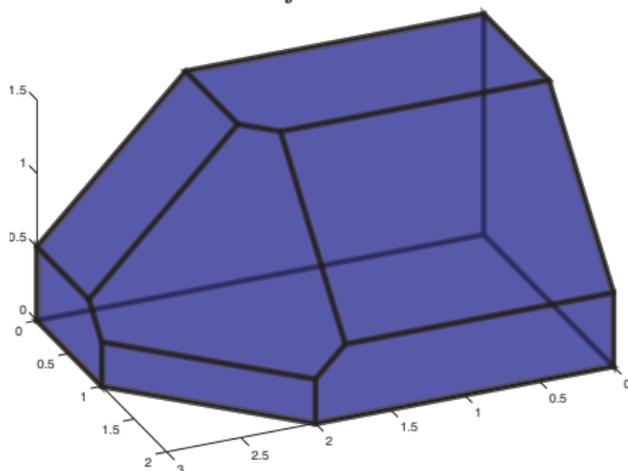
- which axis is which?

Associated polyhedron with a polymatroid function

- Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$,
 $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.

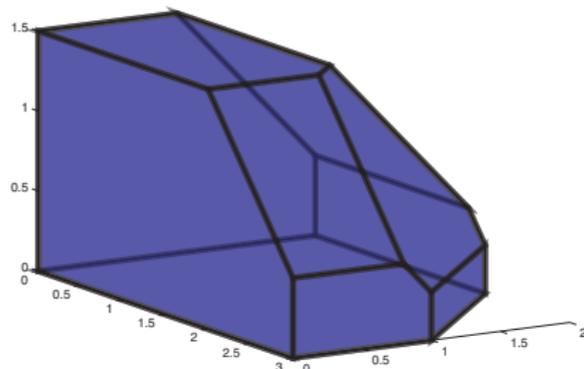
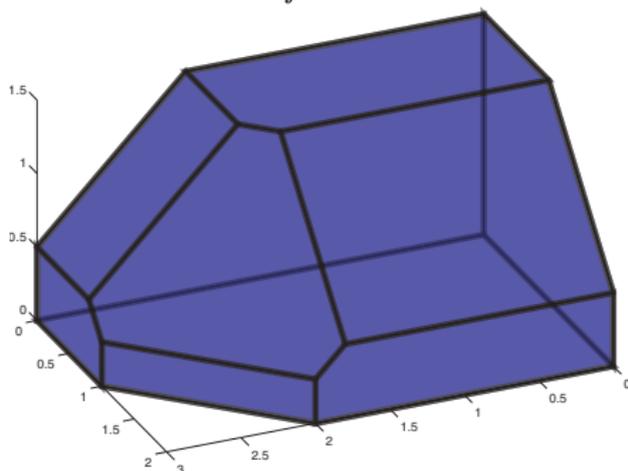
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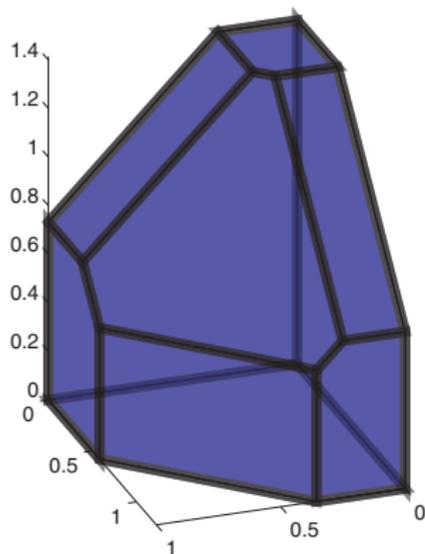
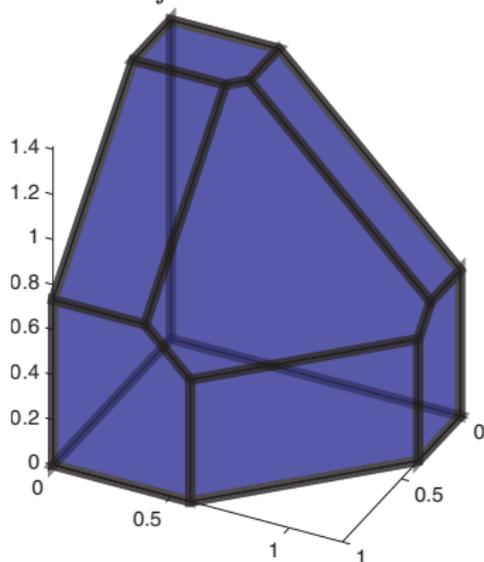
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Associated polyhedron with a polymatroid function

- Consider modular function $w : V \rightarrow \mathbb{R}_+$ as $w = (1, 1.5, 2)^\top$, and then the submodular function $f(S) = \sqrt{w(S)}$.

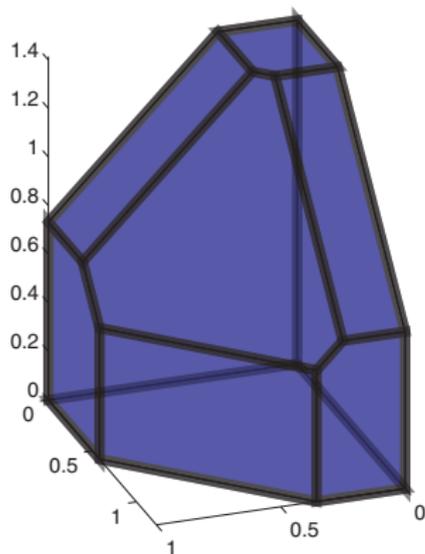
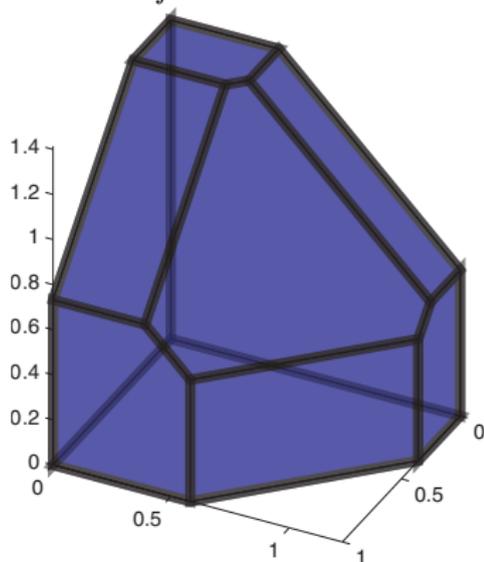
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- which axis is which?

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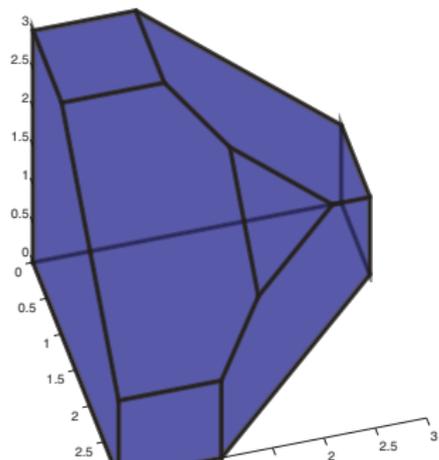
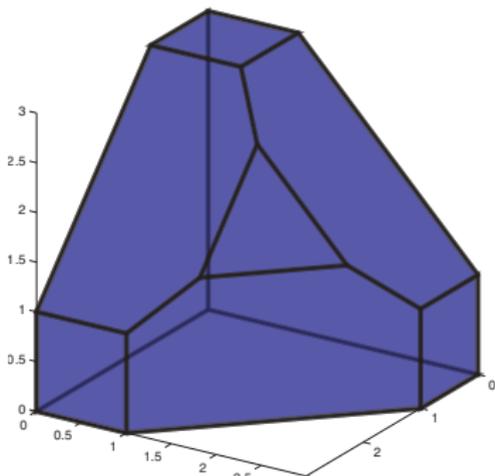
- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular? $f(S) = g(|S|)$ is not submodular since $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5$.

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- Observe: P_f^+ (at two views), maximal independent subvectors not constant rank, hence **not** a polymatroid.



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- Is there any relationship between these two polytopes?
- In the next theorem, we show that any P_f^+ -basis has the same component sum, when f is a polymatroid function, and P_f^+ satisfies the other properties so that P_f^+ is a polymatroid.

A polymatroid function's polyhedron is a polymatroid.

Theorem 10.5.1

Let f be a polymatroid function defined on subsets of E . For any $x \in \mathbb{R}_+^E$, and any P_f^+ -basis $y^x \in \mathbb{R}_+^E$ of x , the component sum of y^x is

$$\begin{aligned} y^x(E) = \text{rank}(x) &\triangleq \max \left(y(E) : y \leq x, y \in P_f^+ \right) \\ &= \min (x(A) + f(E \setminus A) : A \subseteq E) \end{aligned} \quad (10.76)$$

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Taking $E \setminus B = \text{supp}(x)$ (so elements B are all zeros in x), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank} \left(\frac{1}{\epsilon} \mathbf{1}_{E \setminus B} \right) = f(E \setminus B) = \max \left\{ y(E \setminus B) : y \in P_f^+ \right\} \quad (10.77)$$

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In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

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- Doing so will thus establish that P_f^+ is a polymatroid.

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A polymatroid function's polyhedron is a polymatroid.

... proof continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) \leq f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case,

$$\min (x(A) + f(E \setminus A) : A \subseteq E) \quad (10.78)$$

$$= x(E) + \min (f(E \setminus A) - x(E \setminus A) : A \subseteq E) \quad (10.79)$$

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$$= x(E) \quad (10.81)$$

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- This is a value dependent only on x , a self basis, unique P_f^+ -base.

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A polymatroid function's polyhedron is a polymatroid.

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- Then for any order (a_1, a_2, \dots) of the elements and $A_i \triangleq (a_1, a_2, \dots, a_i)$, we have $x(a_i) \geq f(a_i) \geq f(a_i | A_{i-1})$, the second inequality by submodularity.

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$$\min (x(A) + f(E \setminus A) : A \subseteq E) \quad (10.82)$$

$$= x(E) + \min (f(A) - x(A) : A \subseteq E) \quad (10.83)$$

$$= x(E) + \min \left(\sum_i f(a_i|A_{i-1}) - \sum_i x(a_i) : A \subseteq E \right) \quad (10.84)$$

$$= x(E) + \min \left(\sum_i \underbrace{\left(f(a_i|A_{i-1}) - x(a_i) \right)}_{\leq 0} : A \subseteq E \right) \quad (10.85)$$

$$= x(E) + f(E) - x(E) = f(E) = \max(y(E) : y \in P_f^+). \quad (10.86)$$

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- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.

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- For any P_f^+ -basis y^x of x , and any $A \subseteq E$, we have weak relationship:

$$y^x(E) = y^x(A) + y^x(E \setminus A) \quad (10.88)$$

$$\leq x(A) + f(E \setminus A). \quad (10.89)$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$.

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- This ensures

$$\max (y(E) : y \leq x, y \in P_f^+) \leq \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (10.90)$$

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- Given an A where equality in Eqn. (10.89) holds, above min result follows.

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- For any $y \in P_f^+$, call a set $B \subseteq E$ **tight** if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

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- Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.

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- Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.
- For $y \in P_f^+$, it will be ultimately useful to define this lattice family of tight sets: $\mathcal{D}(y) \triangleq \{A : A \subseteq E, y(A) = f(A)\}$.

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- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f , meaning it is an element of some tight set and $e \in \text{sat}(y^x)$ (since if $e \in T \in \mathcal{D}(y^x)$, then $e \in \text{sat}(y^x)$).



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- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).



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- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f , meaning it is an element of some tight set and $e \in \text{sat}(y^x)$ (since if $e \in T \in \mathcal{D}(y^x)$, then $e \in \text{sat}(y^x)$).
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).
- Hence, we have

$$y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \quad (10.95)$$



A polymatroid function's polyhedron is a polymatroid.

... proof continued.

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- So we identified the A to be the elements that are non-tight, and achieved the min, as desired. □

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Theorem 10.5.2

For any polymatroid P (compact subset of \mathbb{R}_+^E , zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \rightarrow \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}_+^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E, y(A) = f(A)\} \quad (10.96)$$

Theorem 10.5.3

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

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We have already proven this as part of Theorem 10.5.1 □

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For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 10.5.1 □

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \quad (10.97)$$

Join \vee and meet \wedge for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \quad (10.98)$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \quad (10.99)$$

Hence,

$$x \vee y \triangleq \left(\max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \dots, \max(x(e_n), y(e_n)) \right)$$

and similarly

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- From this, we can define things like an lattices, and other constructs.