**Submodular Functions, Optimization, and Applications to Machine Learning**  
— Fall Quarter, Lecture 10 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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\[
 f(A) + f(B) \geq f(A \cup B) + f(A \cap B) 
\]

Clockwise from top left:  
Lásló Lovász  
Jack Edmonds  
Satoru Fujishige  
George Nemhauser  
Laurence Wolsey  
András Frank  
Lloyd Shapley  
H. Narayanan  
Robert Bixby  
William Cunningham  
William Tutte  
Richard Rado  
Alexander Schrijver  
Garrett Birkhoff  
Hassler Whitney  
Richard Dedekind

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**Announcements, Assignments, and Reminders**

- Homework 2, due Nov 2nd, 11:59pm on our assignment dropbox  
  (https://canvas.uw.edu/courses/1397085/assignments).
- Final problem on HW2 should now be first problem on HW3 (will be out soon).
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements  
  (https://canvas.uw.edu/courses/1397085/announcements).
- Office hours, Wed & Thur, 10:00pm at our class zoom link.
The greedy algorithm

- In combinatorial optimization, the greedy algorithm is often useful as a heuristic that can work quite well in practice.
- The goal is to choose a good subset of items, and the fundamental tenet of the greedy algorithm is to choose next whatever currently looks best, without the possibility of later recall or backtracking.
- Sometimes, this gives the optimal solution (we saw in Lecture 5 three greedy algorithms that can find the maximum weight spanning tree, namely Kruskal, Jarník/Prim/Dijkstra, and Borůvka’s Algorithms).
- Greedy is good since it can be made to run very fast, e.g., \( O(n \log n) \).
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working optimally is a defining property of a matroid, and is also a defining property of a polymatroid function.
Matroid and the greedy algorithm

- Let \((E, I)\) be an independence system, and we are given a non-negative modular weight function \(w : E \to \mathbb{R}_+\).

**Algorithm 1:** The Matroid Greedy Algorithm

1. Set \(X \leftarrow \emptyset\);
2. while \(\exists v \in E \setminus X\) s.t. \(X \cup \{v\} \in I\) do
3. \(v \in \arg\max \{w(v) : v \in E \setminus X, X \cup \{v\} \in I\}\);
4. \(X \leftarrow X \cup \{v\}\);

- Same as sorting items by decreasing weight \(w\), and then choosing items in that order that retain independence.

**Theorem 10.2.4**

Let \((E, I)\) be an independence system. Then the pair \((E, I)\) is a matroid if and only if for each weight function \(w \in \mathcal{R}_+^E\), Algorithm 1 above leads to a set \(I \in I\) of maximum weight \(w(I)\).

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A normalized monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.
Convex Polytope - key representation theorem

- A polytope can be defined in a number of ways, two of which include

**Theorem 10.2.6**

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- $P$ is the convex hull of a finite set of points.
- If it is a bounded intersection of halfspaces, that is there exits matrix $A$ and vector $b$ such that

$$P = \{ x : Ax \leq b \}$$ (10.9)

- This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carathéodory.

Linear Programming duality forms

There are many ways to construct the dual. For example,

$$\max \{ c^T x | x \geq 0, Ax \leq b \} = \min \{ y^T b | y \geq 0, y^T A \geq c^T \}$$ (10.11)

$$\max \{ c^T x | x \geq 0, Ax = b \} = \min \{ y^T b | y^T A \geq c^T \}$$ (10.12)

$$\min \{ c^T x | x \geq 0, Ax \geq b \} = \max \{ y^T b | y \geq 0, y^T A \leq c^T \}$$ (10.13)

$$\min \{ c^T x | Ax \geq b \} = \max \{ y^T b | y \geq 0, y^T A = c^T \}$$ (10.14)
For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $1_I \in \{0, 1\}^E \subset [0, 1]^E \subset \mathbb{R}_+^E$.

Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\} \subseteq [0, 1]^E \quad (10.1)$$

Now take the rank function $r$ of $M$, and define the following polyhedron:

$$P^+_r \triangleq \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (10.2)$$

Examples of $P^+_r$ are forthcoming.

Now, take any $x \in P_{\text{ind. set}}$, then we will show that that $x \in P^+_r$ (or $P_{\text{ind. set}} \subseteq P^+_r$). We show this after a few examples of $P^+_r$.

Consider this in two dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \quad (10.4)$$

$$x_1 \leq r(\{v_1\}) \in \{0, 1\} \quad (10.5)$$

$$x_2 \leq r(\{v_2\}) \in \{0, 1\} \quad (10.6)$$

$$x_1 + x_2 \leq r(\{v_1, v_2\}) \in \{0, 1, 2\} \quad (10.7)$$

Because $r$ is submodular, we have

$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \quad (10.8)$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either superfluous ($r(v_1, v_2) = r(v_1) + r(v_2)$, “inactive”) or “active.”
Matroid Polyhedron in 2D

And, if $v_2$ is a loop ...

Possible  
Not Possible

Not Possible
Matroid Polyhedron in 3D

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \]  

Consider three dimensions, \( E = \{1, 2, 3\} \). Get equations of the form:

- \( x_1 \geq 0 \) and \( x_2 \geq 0 \) and \( x_3 \geq 0 \)  \hspace{1cm} (10.10)
- \( x_1 \leq r(\{v_1\}) \)  \hspace{1cm} (10.11)
- \( x_2 \leq r(\{v_2\}) \)  \hspace{1cm} (10.12)
- \( x_3 \leq r(\{v_3\}) \)  \hspace{1cm} (10.13)
- \( x_1 + x_2 \leq r(\{v_1, v_2\}) \)  \hspace{1cm} (10.14)
- \( x_2 + x_3 \leq r(\{v_2, v_3\}) \)  \hspace{1cm} (10.15)
- \( x_1 + x_3 \leq r(\{v_1, v_3\}) \)  \hspace{1cm} (10.16)
- \( x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \)  \hspace{1cm} (10.17)

Consider the simple cycle matroid on a graph consisting of a 3-cycle, \( G = (V, E) \) with matroid \( M = (E, \mathcal{I}) \) where \( I \in \mathcal{I} \) is a forest.

- So any set of either one or two edges is independent, and has rank equal to cardinality.
- The set of three edges is dependent, and has rank 2.
Matroid Polyhedron in 3D

Two view of $P_r^+$ associated with a matroid
$(\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\})$.

$P_r^+$ associated with the “free” matroid in 3D.
The next two slides are from the previous lecture.

For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $1_I \in \{0, 1\}^E \subset [0, 1]^E \subset \mathbb{R}_+^E$.

Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\} \subseteq [0, 1]^E \quad (10.1)$$

Now take the rank function $r$ of $M$, and define the following polyhedron:

$$P^+_r \triangleq \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (10.2)$$

Examples of $P^+_r$ are forthcoming.

Now, take any $x \in P_{\text{ind. set}}$, then we will show that that $x \in P^+_r$ (or $P_{\text{ind. set}} \subseteq P^+_r$). We show this after a few examples of $P^+_r$. 
Vector, modular, incidence

- Recall, any vector \( x \in \mathbb{R}^E \) can be seen as a normalized modular function, as for any \( A \subseteq E \), we have
  \[
  x(A) = \sum_{a \in A} x_a \quad (10.11)
  \]

- Given an \( A \subseteq E \), define the incidence vector \( 1_A \in \{0, 1\}^E \) on the unit hypercube as follows:
  \[
  1_A \overset{\text{def}}{=} \left\{ x \in \{0, 1\}^E : x_i = 1 \iff i \in A \right\} \quad (10.12)
  \]
equivalently,
  \[
  1_A(j) \overset{\text{def}}{=} \begin{cases} 
  1 & \text{if } j \in A \\
  0 & \text{if } j \not\in A
  \end{cases} \quad (10.13)
  \]

**Lemma 10.3.1** \((P_{\text{ind. set}} \subseteq P_r^+)\)

- If \( x \in P_{\text{ind. set}} \), then
  \[
  x = \sum_i \lambda_i 1_{I_i} \quad (10.18)
  \]
  for some appropriate vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \).

- Clearly, for such \( x \), \( x \geq 0 \).

- Now, for any \( A \subseteq E \),
  \[
  x(A) = x^T 1_A = \sum_i \lambda_i 1_{I_i}^T 1_A \quad (10.19)
  \]
  \[
  \leq \sum_i \lambda_i \max_{j : I_j \subseteq A} 1_{I_j}(E) \quad (10.20)
  \]
  \[
  = \max_{j : I_j \subseteq A} 1_{I_j}(E) = \max_{I \in \mathcal{I}} |A \cap I| \quad (10.21)
  \]
  \[
  = r(A) \quad (10.22)
  \]
  Thus, \( x \in P_r^+ \) and hence \( P_{\text{ind. set}} \subseteq P_r^+ \).
Thus, we have that:

\[ P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\} \subseteq P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \] (10.23)

Therefore, since \( \{1_I : I \in \mathcal{I}\} \subseteq \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\} = P_{\text{ind. set}} \subseteq P_r^+ \), we have that

\[ \max \{ w(I) : I \in \mathcal{I} \} \leq \max \{ w^T x : x \in P_{\text{ind. set}} \} \leq \max \{ w^T x : x \in P_r^+ \} \] (10.24)

(10.25)

In fact, the two polyhedra \( P_{\text{ind. set}} \) and \( P_r^+ \) are identical (and thus both are polytopes). We’ll show this in the next few theorems.

### Maximum weight independent set via greedy weighted rank

**Theorem 10.3.2**

Let \( M = (V, \mathcal{I}) \) be a matroid, with rank function \( r \), then for any weight function \( w \in \mathbb{R}^V_+ \), there exists a chain of sets \( U_1 \subset U_2 \subset \cdots \subset U_n \subset V \) such that

\[ \max \{ w(I) | I \in \mathcal{I} \} = \sum_{i=1}^{n} \lambda_i r(U_i) \] (10.26)

where \( \lambda_i \geq 0 \) satisfy

\[ w = \sum_{i=1}^{n} \lambda_i 1_{U_i} \] (10.27)
Maximum weight independent set via weighted rank

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

\[
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{pmatrix} = (w_1 - w_2) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \\
\cdots + (w_{n-1} - w_n) \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n) \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}
\]

(10.28)

- If we can take $w$ in non-increasing order ($w_1 \geq w_2 \geq \cdots \geq w_n$), then each coefficient of the vectors is non-negative (except possibly the last one, $w_n$).

Maximum weight independent set via weighted rank

Proof.

- Again assuming $w \in \mathbb{R}^E$, w.l.o.g. order elements of $V$ non-increasing by $w$ so $(v_1, v_2, \ldots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$

- Define the sets $U_i$ based on this order as follows, for $i = 0, \ldots, n$

\[
U_i \overset{\text{def}}{=} \{v_1, v_2, \ldots, v_i\}
\]

(10.29)

- Define the set $I$ as those elements where the rank increases, i.e.:

\[
I \overset{\text{def}}{=} \{v_i | r(U_i) > r(U_{i-1})\}.
\]

(10.30)

Hence, given an $i$ with $v_i \notin I$, $r(U_i) = r(U_{i-1})$.

- Therefore, $I$ is the output of the greedy algorithm for $\max \{w(I) | I \in \mathcal{I}\}$. since items $v_i$ are ordered decreasing by $w(v_i)$, and we only choose the ones that increase the rank, which means they don’t violate independence.

- And therefore, $I$ is a maximum weight independent set (can even be a base, actually).
Maximum weight independent set via weighted rank

Proof.

Now, we define $\lambda_i$ as follows

$$0 \leq \lambda_i \overset{\text{def}}{=} w(v_i) - w(v_{i+1}) \text{ for } i = 1, \ldots, n - 1 \quad (10.31)$$

$$\lambda_n \overset{\text{def}}{=} w(v_n) \quad (10.32)$$

And the weight of the independent set $w(I)$ is given by

$$w(I) = \sum_{v \in I} w(v) = \sum_{i=1}^{n} w(v_i)(r(U_i) - r(U_{i-1}))$$

$$= w(v_n)r(U_n) + \sum_{i=1}^{n-1} (w(v_i) - w(v_{i+1}))r(U_i) = \sum_{i=1}^{n} \lambda_i r(U_i) \quad (10.33)$$

Since we ordered $v_1, v_2, \ldots$ non-increasing by $w$, for all $i$, and since $w \in \mathbb{R}^E_+$, we have $\lambda_i \geq 0$

Linear Program LP

Consider the linear programming primal problem

$$\begin{align*}
\text{maximize} \quad & w^\top x \\
\text{subject to} \quad & x_v \geq 0 \quad (v \in V) \quad (10.35) \\
& x(U) \leq r(U) \quad (\forall U \subseteq V)
\end{align*}$$

And its convex dual (note $y \in \mathbb{R}^{2^n}_+$, $y_U$ is a scalar element within this exponentially big vector):

$$\begin{align*}
\text{minimize} \quad & \sum_{U \subseteq V} y_U r(U), \\
\text{subject to} \quad & y_U \geq 0 \quad (\forall U \subseteq V) \\
& \sum_{U \subseteq V} y_U 1_U \geq w
\end{align*} \quad (10.36)$$

Thanks to strong duality, the solutions to these are equal to each other.
Consider the linear programming primal problem

\[
\begin{align*}
\text{maximize} & \quad w^\top x \\
\text{s.t.} & \quad x_v \geq 0 \quad (v \in V) \\
& \quad x(U) \leq r(U) \quad (\forall U \subseteq V)
\end{align*}
\] (10.37)

This is identical to the problem

\[
\begin{align*}
\text{max } w^\top x \text{ such that } x \in P_{\text{ind. set}}
\end{align*}
\] (10.38)

where, again, \( P_{\text{ind. set}} \subseteq P_r^+ \), the above problem can only have a larger solution. I.e.,

\[
\begin{align*}
\text{max } w^\top x \text{ s.t. } x \in P_{\text{ind. set}} \leq \text{max } w^\top x \text{ s.t. } x \in P_r^+.
\end{align*}
\] (10.39)

Hence, we have the following relations:

\[
\begin{align*}
\max \{w(I) : I \in \mathcal{I}\} & \leq \text{max } \{w^\top x : x \in P_{\text{ind. set}}\} \\
& \leq \text{max } \{w^\top x : x \in P_r^+\}
\end{align*}
\] (10.40)

\[
\begin{align*}
def \alpha_{\text{min}} = \min \left\{ \sum_{U \subseteq V} y_U r(U) : \forall U, y_U \geq 0; \sum_{U \subseteq V} y_U 1_U \geq w \right\}
\end{align*}
\] (10.42)

Theorem 10.3.2 states that

\[
\begin{align*}
\max \{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i)
\end{align*}
\] (10.43)

for the chain of \( U_i \)'s and \( \lambda_i \geq 0 \) that satisfies \( w = \sum_{i=1}^{n} \lambda_i 1_{U_i} \) (i.e., the r.h.s. of Eq. 10.43 is feasible w.r.t. the dual LP).

Therefore, we also have \( \max \{w(I) : I \in \mathcal{I}\} \leq \alpha_{\text{min}} \) and

\[
\begin{align*}
\max \{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \geq \alpha_{\text{min}}
\end{align*}
\] (10.44)
Polytope equivalence

Hence, we have the following relations:
\[
\max \{w(I) : I \in \mathcal{I}\} = \max \{w^\top x : x \in \text{P}_{\text{ind. set}}\} = \max \{w^\top x : x \in P_r^+\} \quad (10.40)
\]
\[
\text{def} \quad \alpha_{\text{min}} = \min \left\{ \sum_{U \subseteq V} y_U r(U) : \forall U, y_U \geq 0; \sum_{U \subseteq V} y_U 1_U \geq w \right\} \quad (10.42)
\]

Therefore, all the inequalities above are equalities.

And since \( w \in \mathbb{R}_E^+ \) is an arbitrary direction into the positive orthant, we see that \( P_r^+ = P_{\text{ind. set}} \).

That is, we have just proven:

**Theorem 10.3.3**

\[
P_r^+ = P_{\text{ind. set}} \quad (10.45)
\]

Polytope Equivalence (Summarizing the above)

- For each \( I \in \mathcal{I} \) of a matroid \( M = (E, \mathcal{I}) \), we can form the incidence vector \( 1_I \).
- Taking the convex hull, we get the independent set polytope, that is
  \[
P_{\text{ind. set}} = \text{conv} \{ \bigcup_{I \in \mathcal{I}} \{1_I\} \} \quad (10.46)
  \]
- Now take the rank function \( r \) of \( M \), and define the following polytope:
  \[
P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (10.47)
  \]

**Theorem 10.3.4**

\[
P_r^+ = P_{\text{ind. set}} \quad (10.48)
\]
Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq 10.40, the LP problem with exponential number of constraints \( \max \{ w^\top x : x \in P^+_r \} \) is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

**Theorem 10.3.5**

The LP problem \( \max \{ w^\top x : x \in P^+_r \} \) can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since \( P^+_r \) is described as the intersection of an exponential number of half spaces).
- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

Base Polytope Equivalence

- Consider convex hull of indicator vectors just of the bases of a matroid, rather than all of the independent sets.
- Consider a polytope defined by the following constraints:

\[
x \geq 0 \\
x(A) \leq r(A) \, \forall A \subseteq V \\
x(V) = r(V)
\]  

(10.49) \hspace{1cm} (10.50) \hspace{1cm} (10.51)

- Note the third requirement, \( x(V) = r(V) \).
- By essentially the same argument as above (Exercise:), we can shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 10.49- 10.51 above.
- What does this look like? The base polytope.
**Spanning set polytope**

- Recall, a set $A$ is spanning in a matroid $M = (E, I)$ if $r(A) = r(E)$.
- Consider convex hull of incidence vectors of spanning sets of a matroid $M$, and call this $P_{\text{spanning}}(M)$.

**Theorem 10.3.6**

*The spanning set polytope is determined by the following equations:*

\[
\begin{align*}
0 \leq x_e & \leq 1 \quad \text{for } e \in E \quad (10.52) \\
x(A) \geq r(E) - r(E \setminus A) & \quad \text{for } A \subseteq E \quad (10.53)
\end{align*}
\]

- Example of spanning set polytope in 2D.

\[
x_1 + x_2 = r(\{v_1, v_2\}) = 1
\]

**Proof.**

- Recall that any $A$ is spanning in $M$ iff $E \setminus A$ is independent in $M^*$ (the dual matroid).
- For any $x \in \mathbb{R}^E$, we have that

\[
x \in P_{\text{spanning}}(M) \iff 1 - x \in P_{\text{ind. set}}(M^*) \quad (10.54)
\]

as we show next . . .
Spanning set polytope

... proof continued.
- This follows since if \( x \in P_{\text{spanning}}(M) \), we can represent \( x \) as a convex combination:
  \[
x = \sum_{i} \lambda_i 1_{A_i}(10.55)
  \]
  where \( A_i \) is spanning in \( M \).
- Consider
  \[
  1 - x = 1_E - x = 1_E - \sum_i \lambda_i 1_{A_i} = \sum_i \lambda_i 1_{E \setminus A_i},
  \]
  which follows since \( \sum_i \lambda_i 1 = 1_E \), so \( 1 - x \) is a convex combination of independent sets in \( M^* \) and so \( 1 - x \in P_{\text{ind. set}}(M^*) \).

... proof continued.
- which means, from the definition of \( P_{\text{ind. set}}(M^*) \), that
  \[
  1 - x \geq 0 \quad(10.57)
  \]
  \[
  1_A - x(A) = |A| - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E
  \]
  And we know the dual rank function is
  \[
  r_{M^*}(A) = |A| + r_{M}(E \setminus A) - r_{M}(E)
  \]
- giving
  \[
  x(A) \geq r_{M}(E) - r_{M}(E \setminus A) \text{ for all } A \subseteq E
  \]
Matroids
where are we going with this?

- We've been discussing results about matroids (independence polytope, etc.).
- By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it is worth showing a result for a general submodular function first.
- Henceforth, we will skip between submodular functions and matroids, each lecture talking less about matroids specifically and taking more about submodular functions more generally ...

Maximal points in a set

- Regarding sets, a subset $X$ of $S$ is a maximal subset of $S$ possessing a given property $\Psi$ if $X$ possesses property $\Psi$ and no set properly containing $X$ (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\Psi$.
- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector $x$ is maximal within $P$ if it is the case that for any $\epsilon > 0$, and for all directions $e \in E$, we have that

$$x + \epsilon \mathbf{1}_e \notin P$$  \hspace{1cm} (10.61)

- Examples of maximal regions (in red)
Maximal points in a set

- Regarding sets, a subset $X$ of $S$ is a **maximal** subset of $S$ possessing a given property $P$ if $X$ possesses property $P$ and no set properly containing $X$ (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $P$.

- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector $x$ is **maximal within** $P$ if it is the case that for any $\epsilon > 0$, and for all directions $e \in E$, we have that

$$x + \epsilon \mathbf{1}_e \notin P$$  \hspace{1cm} (10.61)

- Examples of non-maximal regions (in green)

![Diagram of non-maximal regions](image)

Review from Lecture 6

- The next slide comes from Lecture 6.
Matroids, independent sets, and bases

- **Independent sets**: Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called **independent** if $A \in \mathcal{I}$ and otherwise $A$ is called **dependent**.

- **A base of $U \subseteq E$**: For $U \subseteq E$, a subset $B \subseteq U$ is called a **base** of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

- **A base of a matroid**: If $U = E$, then a “base of $E$” is just called a **base** of the matroid $M$ (this corresponds to a **basis** in a linear space, or a **spanning forest** in a graph, or a **spanning tree** in a connected graph).

---

**$P$-basis of $x$ given compact set $P \subseteq \mathbb{R}_+^E$**

**Definition 10.4.1 (subvector)**

$y$ is a **subvector** of $x$ if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

**Definition 10.4.2 ($P$-basis)**

Given a compact set $P \subseteq \mathbb{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector $y$ of $x$ is called a **$P$-basis** of $x$ if $y$ maximal in $P$.

In other words, $y$ is a $P$-basis of $x$ if $y$ is a maximal $P$-contained subvector of $x$.

Here, by $y$ being “maximal”, we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon 1_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of $y$ (the properties of $y$ being: in $P$, and a subvector of $x$).

In still other words: $y$ is a $P$-basis of $x$ if:

1. $y \leq x$ ($y$ is a subvector of $x$); and
2. $y \in P$ and $y + \epsilon 1_e \notin P$ for all $e \in E$ where $y(e) < x(e)$ and $\forall \epsilon > 0$ ($y$ is maximal $P$-contained).
A vector form of rank

- Recall the definition of rank from a matroid \( M = (E, \mathcal{I}) \).
  \[
  \text{rank}(A) = \max \{|I| : I \subseteq A, I \in \mathcal{I}\} = \max_{I \in \mathcal{I}} |A \cap I| \quad (10.62)
  \]

- **vector rank:** Given a compact set \( P \subseteq \mathbb{R}^E_+ \), define a form of “vector rank” relative to \( P \): Given an \( x \in \mathbb{R}^E \):
  \[
  \text{rank}(x) = \max \{y(E) : y \leq x, y \in P\} = \max_{y \in P} (x \wedge y)(E) \quad (10.63)
  \]

where \( y \leq x \) is componentwise inequality \((y_i \leq x_i, \forall i)\), and where \((x \wedge y) \in \mathbb{R}^E_+\) has \((x \wedge y)(i) = \min(x(i), y(i))\).

- Sometimes use \( \text{rank}_P(x) \) to make \( P \) explicit.
- If \( B_x \) is the set of \( P \)-bases of \( x \), then \( \text{rank}(x) = \max_{y \in B_x} y(E) \).
- If \( x \in P \), then \( \text{rank}(x) = x(E) \) (\( x \) is its own unique self \( P \)-basis).
- If \( x_{\min} \in \argmin_{x \in P} x(E) \), and \( x \leq x_{\min} \) what then? Then \( \text{rank}(x) \) is either \( x(E) \) (if \( x = x_{\min} \)) or otherwise \( \text{rank}(x) = -\infty \).
- In general, might be hard to compute and/or have ill-defined properties.

Next, we look at an object that restrains and cultivates this form of rank.

Polymatroidal polyhedron (or a “polymatroid”)

**Definition 10.4.3** (polymatroid)

A **polymatroid** is a compact set \( P \subseteq \mathbb{R}^E_+ \) satisfying

1. \( 0 \in P \)
2. If \( y \leq x \in P \) then \( y \in P \) (called **down monotone**).
3. For every \( x \in \mathbb{R}^E_+ \), any maximal vector \( y \in P \) with \( y \leq x \) (i.e., any \( P \)-basis of \( x \)), has the same component sum \( y(E) \)

- Condition 3 restated: That is for any two distinct maximal vectors \( y^1, y^2 \in P \), with \( y^1 \leq x \) & \( y^2 \leq x \), with \( y^1 \neq y^2 \), we must have \( y^1(E) = y^2(E) \).
- Condition 3 restated (again): For every vector \( x \in \mathbb{R}^E_+ \), every maximal independent (i.e., \( \in P \)) subvector \( y \) of \( x \) has the same component sum \( y(E) = \text{rank}(x) \).
- Condition 3 restated (yet again): All \( P \)-bases of \( x \) have the same component sum.
Polymatroidal polyhedron (or a “polymatroid”)

Definition 10.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}^E_+$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}^E_+$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any $P$-basis of $x$), has the same component sum $y(E)$

- Vectors within $P$ (i.e., any $y \in P$) are called independent, and any vector outside of $P$ is called dependent.
- Since all $P$-bases of $x$ have the same component sum, if $B_x$ is the set of $P$-bases of $x$, than rank$(x) = y(E)$ for any $y \in B_x$.

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, I)$
2. empty-set containing $\emptyset \in I$
3. down closed, $\emptyset \subseteq I' \subseteq I \in I \Rightarrow I' \in I$.
4. any maximal set $I$ in $I$, bounded by another set $A$, has the same matroid rank (any maximal independent subset $I \subseteq A$ has same size $|I|$).

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}^E_+$
2. zero containing, $0 \in P$
3. down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$
4. any maximal vector $y$ in $P$, bounded by another vector $x$, has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).
Polymatroidal polyhedron (or a “polymatroid”)

Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 = \text{const.}$
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such $y$ must have the same value $y(E)$.
- On the right, there is only one maximal $y \in P$. Since there is only one, the condition on the same value of $y(E)$, $\forall y$ is vacuous.

$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then $x$ is its own $P$-basis, i.e., it is a self $P$-basis.
- In a matroid, a base of $A$ is the maximally contained independent set. If $A$ is already independent, then $A$ is a self-base of $A$ (as we saw in previous Lectures)
Polymatroid as well? no

Left and right: \( \exists \) multiple maximal \( y \leq x \) as indicated.

- On the left, we see there are multiple possible maximal such \( y \in P \) that are \( y \leq x \). Each such \( y \) must have the same value \( y(E) \), but since the equation for the curve is \( y_1^2 + y_2^2 = \text{const.} \neq y_1 + y_2 \), we see this is not a polymatroid.
- On the right, we have a similar situation, just the set of potential values that must have the \( y(E) \) condition changes, but the values of course are still not constant.

Other examples: Polymatroid or not?
It appears that we have five possible forms of polymatroid in 2D, when neither of the elements \{v_1, v_2\} are self-dependent.

- On the left: full dependence between \(v_1\) and \(v_2\)
- Next: full independence between \(v_1\) and \(v_2\)
- Next: partial independence between \(v_1\) and \(v_2\)
- Right two: other forms of partial independence between \(v_1\) and \(v_2\)
  - The \(P\)-bases (or single \(P\)-base in the middle case) are as indicated.
  - Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.
  - The set of \(P\)-bases for a polytope is called the base polytope.

Note that if \(x\) contains any zeros (i.e., suppose that \(x \in \mathbb{R}^E_+\) has \(E \setminus S\) s.t. \(x(E \setminus S) = 0\), so \(S\) indicates the non-zero elements, or \(S = \text{supp}(x)\)), then this also forces \(y(E \setminus S) = 0\), so that \(y(E) = y(S)\). This is true either for \(x \in P\) or \(x \notin P\).

Therefore, in this case, it is the non-zero elements of \(x\), corresponding to elements \(S\) (i.e., the support \(\text{supp}(x)\) of \(x\)), determine the common component sum.

For the case of either \(x \notin P\) or right at the boundary of \(P\), we might give a “name” to this component sum, lets say \(f(S)\) for any given set \(S\) of non-zero elements of \(x\). We could name \(\text{rank}(1_{\epsilon}S) \triangleq f(S)\) for \(\epsilon\) small enough. What kind of function might \(f\) be?
Polymatroid function and its polyhedron.

Definition 10.4.4

A **polymatroid function** is a real-valued function $f$ defined on subsets of $E$ which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron $P^+_f$ associated with a polymatroid function as follows

$$P^+_f = \{ y \in \mathbb{R}^E_+ : y(A) \leq f(A) \text{ for all } A \subseteq E \}$$

(10.64)

$$= \{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \}$$

(10.65)

Associated polyhedron with a polymatroid function

$$P^+_f = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \}$$

(10.66)

Consider this in three dimensions. We have equations of the form:

- $x_1 \geq 0$ and $x_2 \geq 0$ and $x_3 \geq 0$ 
- $x_1 \leq f(\{v_1\})$ 
- $x_2 \leq f(\{v_2\})$ 
- $x_3 \leq f(\{v_3\})$ 
- $x_1 + x_2 \leq f(\{v_1, v_2\})$ 
- $x_2 + x_3 \leq f(\{v_2, v_3\})$ 
- $x_1 + x_3 \leq f(\{v_1, v_3\})$ 
- $x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\})$
Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph \( v_1 - v_2 - v_3 \). That is, \( f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}| \) is count of any edges within \( S \) or between \( S \) and \( V \setminus S \), so that \( \delta(S) = f(S) + f(V \setminus S) - f(V) \) is the standard graph cut.
- Observe: \( P_f^+ \) (at two views):

which axis is which?

Associated polyhedron with a polymatroid function

- Consider: \( f(\emptyset) = 0, f(\{v_1\}) = 1.5, f(\{v_2\}) = 2, f(\{v_1, v_2\}) = 2.5, f(\{v_3\}) = 3, f(\{v_3, v_1\}) = 3.5, f(\{v_3, v_2\}) = 4, f(\{v_3, v_2, v_1\}) = 4.3. \)
- Observe: \( P_f^+ \) (at two views):

which axis is which?
Associated polyhedron with a polymatroid function

- Consider modular function \( w : V \to \mathbb{R}_+ \) as \( w = (1, 1.5, 2)^\top \), and then the submodular function \( f(S) = \sqrt{w(S)} \).
- Observe: \( P_f^+ \) (at two views):

![Diagram of associated polyhedron]

which axis is which?

Associated polytope with a non-submodular function

- Consider function on integers: \( g(0) = 0, g(1) = 3, g(2) = 4, \) and \( g(3) = 5.5 \). Is \( f(S) = g(|S|) \) submodular? \( f(S) = g(|S|) \) is not submodular since \( f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8 \) but \( f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5 \). Alternatively, consider concavity violation, \( 1 = g(1+1) - g(1) < g(2+1) - g(2) = 1.5 \).
- Observe: \( P_f^+ \) (at two views), maximal independent subvectors not constant rank, hence not a polymatroid.

![Diagram of associated polytope]
A polymatroid vs. a polymatroid function’s polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function $f$, its associated polytope is given as
    \[ P_f^+ = \{ y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E \} \]  
    (10.75)
  - We also have the definition of a polymatroidal polytope $P$ (compact subset, zero containing, down-monotone, and $\forall x$ any maximal independent subvector $y \leq x$ has same component sum $y(E)$).
  - Is there any relationship between these two polytopes?
  - In the next theorem, we show that any $P_f^+$-basis has the same component sum, when $f$ is a polymatroid function, and $P_f^+$ satisfies the other properties so that $P_f^+$ is a polymatroid.

Theorem 10.5.1

Let $f$ be a polymatroid function defined on subsets of $E$. For any $x \in \mathbb{R}_+^E$, and any $P_f^+$-basis $y^x \in \mathbb{R}_+^E$ of $x$, the component sum of $y^x$ is

\[
y^x(E) = \text{rank}(x) \triangleq \max \left( y(E) : y \leq x, y \in P_f^+ \right) = \min \left( x(A) + f(E \setminus A) : A \subseteq E \right)
\]

(10.76)

As a consequence, $P_f^+$ is a polymatroid, since r.h.s. is constant w.r.t. $y^x$.

Taking $E \setminus B = \text{supp}(x)$ (so elements $B$ are all zeros in $x$), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

\[
\text{rank} \left( \frac{1}{\epsilon} \mathbf{1}_{E \setminus B} \right) = f(E \setminus B) = \max \left\{ y(E \setminus B) : y \in P_f^+ \right\}
\]

(10.77)

In fact, we will ultimately see a number of important consequences of this theorem (other than just that $P_f^+$ is a polymatroid).
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since $f$ is non-negative.
- Also, for any $y \in P_f^+$ then any $x \leq y$ is also such that $x \in P_f^+$. So, $P_f^+$ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., $y^x$ is a $P_f^+$-basis of $x$).
- Goal is to show that any such $y^x$ has $y^x(E) = \text{const}$, dependent only on $x$ and also $f$ (which defines the polytope) but not dependent on $y^x$, the particular $P_f^+$-basis.
- Doing so will thus establish that $P_f^+$ is a polymatroid.

... proof continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) \leq f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case,

  $$
  \min (x(A) + f(E \setminus A) : A \subseteq E) = x(E) + \min (f(E \setminus A) - x(E \setminus A) : A \subseteq E)
  $$

  \ begun{align}
  &= x(E) + \min (f(A) - x(A) : A \subseteq E) \\
  &= x(E)
  \end{align}

  (10.78) \hspace{0.5cm} (10.79) \hspace{0.5cm} (10.80) \hspace{0.5cm} (10.81)

- When $x \in P_f^+$, $y = x$ is clearly the solution to

  $$
  \max \left(y(E) : y \leq x, y \in P_f^+\right), \text{ so this is tight, and rank}(x) = x(E).
  $$

- This is a value dependent only on $x$, a self basis, unique $P_f^+$-base.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- 2nd trivial case: \( x(A) > f(A), \forall A \subseteq E \) (i.e., \( x \notin P^+_f \) every direction).
- Then for any order \((a_1, a_2, \ldots)\) of the elements and \( A_i \triangleq (a_1, a_2, \ldots, a_i) \), we have \( x(a_i) \geq f(a_i) \geq f(a_i | A_{i-1}) \), the second inequality by submodularity. This gives

\[
\min (x(A) + f(E \setminus A) : A \subseteq E) = x(E) + \min (f(A) - x(A) : A \subseteq E) \tag{10.82}
\]

\[
= x(E) + \min \left( \sum_i f(a_i | A_{i-1}) - \sum_i x(a_i) : A \subseteq E \right) \tag{10.83}
\]

\[
= x(E) + \min \left( \sum_i \left( f(a_i | A_{i-1}) - x(a_i) \right) : A \subseteq E \right) \tag{10.84}
\]

\[
= x(E) + f(E) - x(E) = f(E) = \max (y(E) : y \in P^+_f).\tag{10.85}
\]

A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Assume neither trivial case. Because \( y^x \in P^+_f \), we have that \( y^x(A) \leq f(A) \) for all \( A \subseteq E \).
- We show that the constant is given by

\[
y^x(E) = \min (x(A) + f(E \setminus A) : A \subseteq E) \tag{10.87}
\]

- For any \( P^+_f \)-basis \( y^x \) of \( x \), and any \( A \subseteq E \), we have weak relationship:

\[
y^x(E) = y^x(A) + y^x(E \setminus A) \tag{10.88}
\]

\[
\leq x(A) + f(E \setminus A). \tag{10.89}
\]

This follows since \( y^x \leq x \) and since \( y^x \in P^+_f \).

- This ensures

\[
\max \left( y(E) : y \leq x, y \in P^+_f \right) \leq \min (x(A) + f(E \setminus A) : A \subseteq E) \tag{10.90}
\]

- Given an \( A \) where equality in Eqn. (10.89) holds, above min result follows.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- For any $y \in P^+_f$, call a set $B \subseteq E$ **tight** if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

\[
    f(B) + f(C) = y(B) + y(C)
\]

(10.91)

\[
    = y(B \cap C) + y(B \cup C) 
\]

(10.92)

\[
    \leq f(B \cap C) + f(B \cup C)
\]

(10.93)

\[
    \leq f(B) + f(C)
\]

(10.94)

which requires equality everywhere above.

- Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.

- For $y \in P^+_f$, it will be ultimately useful to define this lattice family of tight sets: $D(y) \triangleq \{A : A \subseteq E, y(A) = f(A)\}$.

Also, we define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{T : T \in D(y)\}$, so $y(\text{sat}(y)) = f(\text{sat}(y))$.

- Consider again a $P^+_f$-basis $y^x$ (so maximal).

- Given a $e \in E$, either $y^x(e)$ is cut off due to $x$ (so $y^x(e) = x(e)$) or $e$ is saturated by $f$, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$ (since if $e \in T \in D(y^x)$, then $e \in \text{sat}(y^x)$).

- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).

- Hence, we have

\[
    y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A)
\]

(10.95)

- So we identified the $A$ to be the elements that are non-tight, and achieved the min, as desired.
A polymatroid is a polymatroid function’s polytope

- So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.
- Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P_f^+$?

**Theorem 10.5.2**

For any polymatroid $P$ (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \}$.

**Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$**

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, y(A) = f(A) \}$$  \hspace{1cm} (10.96)

**Theorem 10.5.3**

For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

**Proof.**

We have already proven this as part of Theorem 10.5.1  □

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$:

$$\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \}$$  \hspace{1cm} (10.97)
Join $\lor$ and meet $\land$ for $x, y \in \mathbb{R}^E_+$

- For $x, y \in \mathbb{R}^E_+$, define vectors $x \land y \in \mathbb{R}^E_+$ and $x \lor y \in \mathbb{R}^E_+$ such that, for all $e \in E$

  \[
  (x \land y)(e) = \min(x(e), y(e)) \quad (10.99)
  \]

  \[
  (x \lor y)(e) = \max(x(e), y(e)) \quad (10.98)
  \]

  Hence,

  \[
  x \lor y \triangleq \left( \max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \ldots, \max(x(e_n), y(e_n)) \right)
  \]

  and similarly

  \[
  x \land y \triangleq \left( \min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \ldots, \min(x(e_n), y(e_n)) \right)
  \]

- From this, we can define things like an lattices, and other constructs.