Submodular Functions, Optimization, and Applications to Machine Learning
— Fall Quarter, Lecture 10 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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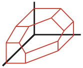
Nov 2nd, 2020

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad$ $= f(A_r) + f(C) + f(B_r)$ $\quad$ $= f(A \cap B)$



Announcements, Assignments, and Reminders

- Homework 2, due Nov 2nd, 11:59pm on our assignment dropbox (`https://canvas.uw.edu/courses/1397085/assignments`).
- Final problem on HW2 should now be first problem on HW3 (will be out soon).
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (`https://canvas.uw.edu/courses/1397085/announcements`).
- Office hours, Wed & Thur, 10:00pm at our class zoom link.

Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid
- L9(10/28): Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy, Polyhedra, Matroid Polytopes
- L10(11/2): Matroid Polytopes, Matroids $\rightarrow$ Polymatroids, Polymatroids
- L11(11/4):
- L12(11/9):
- L–(11/11): Veterans Day, Holiday
- L13(11/16):
- L14(11/18):
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

The greedy algorithm

- In combinatorial optimization, the greedy algorithm is often useful as a heuristic that can work quite well in practice.
- The goal is to choose a good subset of items, and the fundamental tenet of the greedy algorithm is to choose next whatever currently looks best, without the possibility of later recall or backtracking.
- Sometimes, this gives the optimal solution (we saw in Lecture 5 three greedy algorithms that can find the maximum weight spanning tree, namely Kruskal, Jarník/Prim/Dijkstra, and Borůvka's Algorithms).
- Greedy is good since it can be made to run very fast, e.g., $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working optimally is a defining property of a matroid, and is also a defining property of a polymatroid function.

Matroid and the greedy algorithm

- Let $(E, \mathcal{I})$ be an independence system, and we are given a non-negative modular weight function $w : E \to \mathbb{R}_+$.

Algorithm 1: The Matroid Greedy Algorithm

1 Set $X \leftarrow \emptyset$;
2 **while** $\exists v \in E \setminus X$ *s.t.* $X \cup \{v\} \in \mathcal{I}$ **do**
3 $\quad v \in \operatorname{argmax} \{w(v) : v \in E \setminus X,\ X \cup \{v\} \in \mathcal{I}\}$;
4 $\quad X \leftarrow X \cup \{v\}$;

- Same as sorting items by decreasing weight w, and then choosing items in that order that retain independence.

Theorem 10.2.4

Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm ?? above leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A normalized monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.

Convex Polytope - key representation theorem

- A polytope can be defined in a number of ways, two of which include

Theorem 10.2.6

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- *P is the convex hull of a finite set of points.*
- *If it is a bounded intersection of halfspaces, that is there exits matrix A and vector b such that*

$$P = \{x : Ax \leq b\} \tag{10.9}$$

- This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carátheodory.

Linear Programming duality forms

There are many ways to construct the dual. For example,

$$\max\{c^\intercal x | x \geq 0, Ax \leq b\} = \min\{y^\intercal b | y \geq 0, y^\intercal A \geq c^\intercal\} \tag{10.11}$$

$$\max\{c^\intercal x | x \geq 0, Ax = b\} = \min\{y^\intercal b | y^\intercal A \geq c^\intercal\} \tag{10.12}$$

$$\min\{c^\intercal x | x \geq 0, Ax \geq b\} = \max\{y^\intercal b | y \geq 0, y^\intercal A \leq c^\intercal\} \tag{10.13}$$

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Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I \in \{0,1\}^E \subset [0,1]^E \subset \mathbb{R}_+^E$.

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- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \subseteq [0,1]^E \tag{10.1}$$

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- Now take the rank function r of M, and define the following polyhedron:

$$P_r^+ \triangleq \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \tag{10.2}$$

Examples of P_r^+ are forthcoming.

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Matroid Polyhedron in 2D

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{10.3}$$

- Consider this in two dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \tag{10.4}$$

$$x_1 \leq r(\{v_1\}) \in \{0, 1\} \tag{10.5}$$

$$x_2 \leq r(\{v_2\}) \in \{0, 1\} \tag{10.6}$$

$$x_1 + x_2 \leq r(\{v_1, v_2\}) \in \{0, 1, 2\} \tag{10.7}$$

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$$x_1 + x_2 \leq r(\{v_1, v_2\}) \in \{0, 1, 2\} \tag{10.7}$$

- Because r is submodular, we have

$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \tag{10.8}$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either superfluous ($r(v_1, v_2) = r(v_1) + r(v_2)$, "inactive") or "active."

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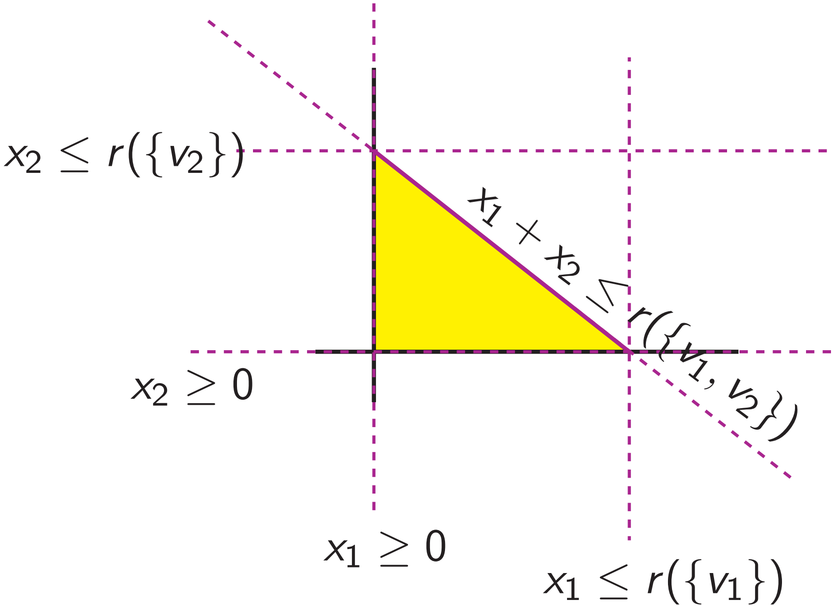

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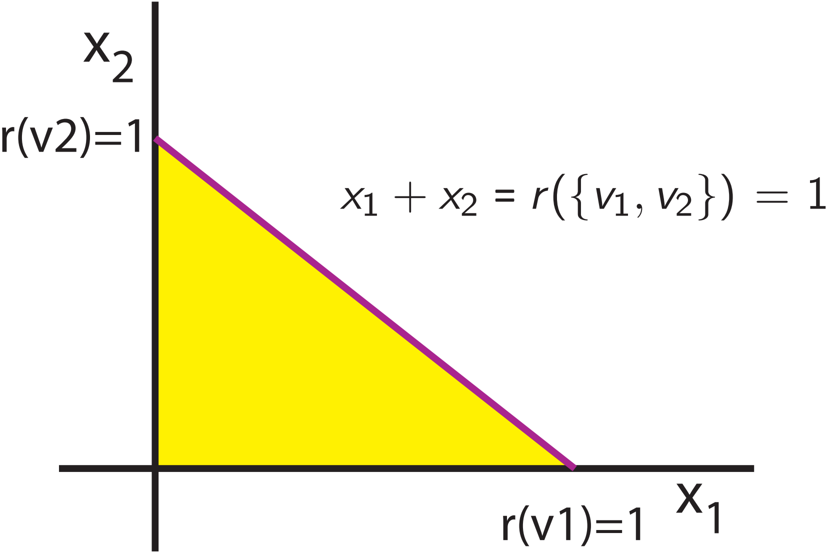

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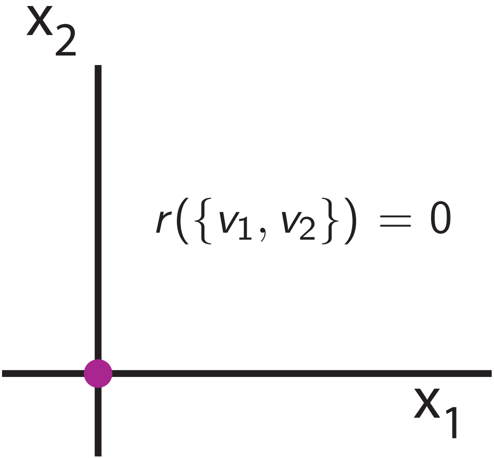

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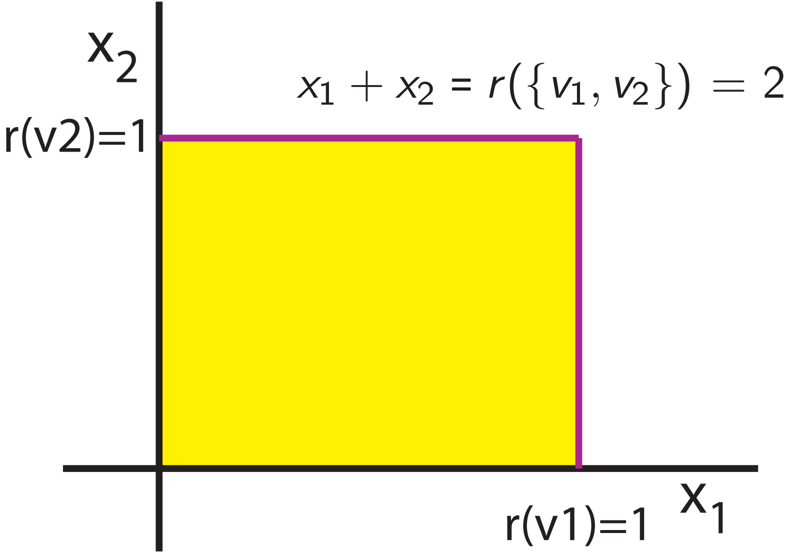

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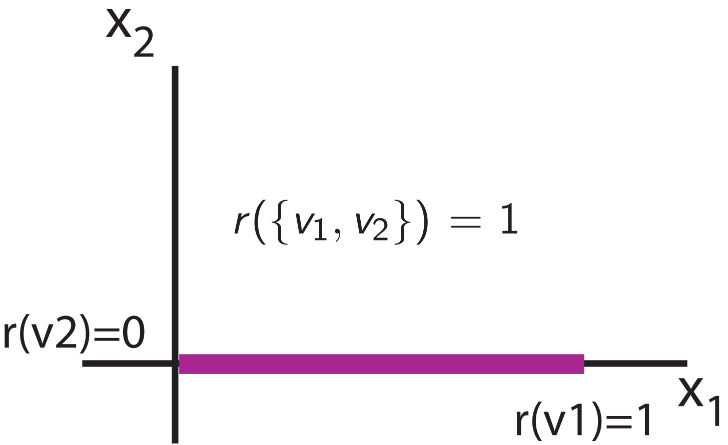
And, if v2 is a loop ...



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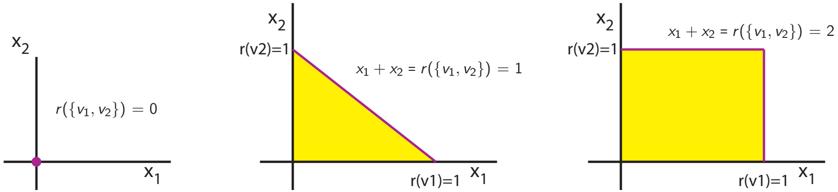

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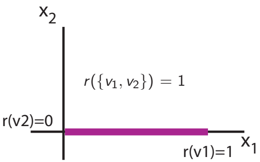

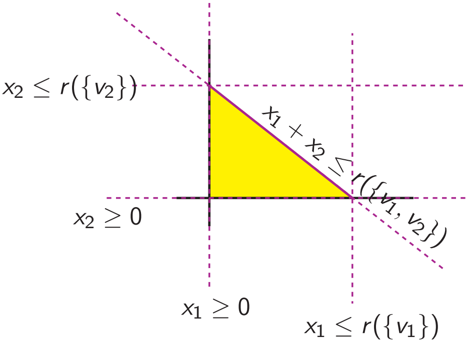

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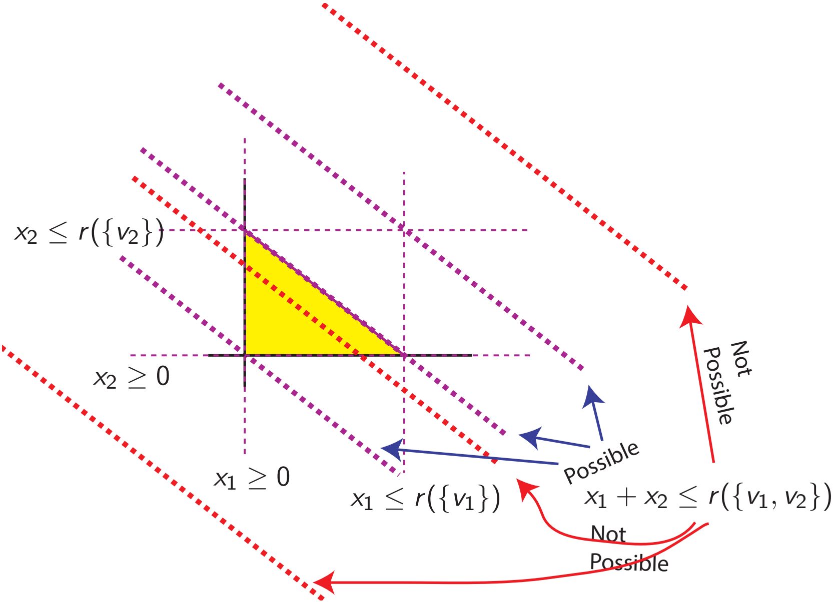

Matroid Polyhedron in 3D

$$P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \tag{10.9}$$

- Consider three dimensions, $E = \{1, 2, 3\}$. Get equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \tag{10.10}$$

$$x_1 \leq r(\{v_1\}) \tag{10.11}$$

$$x_2 \leq r(\{v_2\}) \tag{10.12}$$

$$x_3 \leq r(\{v_3\}) \tag{10.13}$$

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$$x_1 + x_3 \leq r(\{v_1, v_3\}) \tag{10.16}$$

$$x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \tag{10.17}$$

Matroid Polyhedron in 3D

- Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.

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- So any set of either one or two edges is independent, and has rank equal to cardinality.

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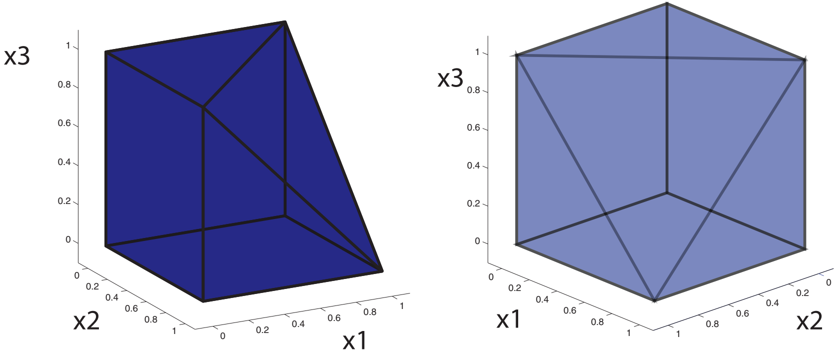
- So any set of either one or two edges is independent, and has rank equal to cardinality.
- The set of three edges is dependent, and has rank 2.

Matroid Polyhedron in 3D

Two view of P_r^+ associated with a matroid $(\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\})$.



Matroid Polyhedron in 3D

P_r^+ associated with the "free" matroid in 3D.

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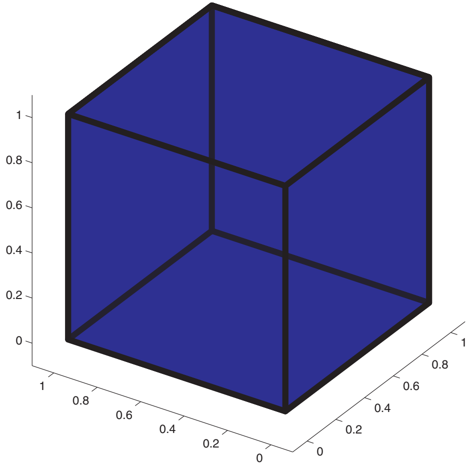

Review

- The next two slides are from the previous lecture.

Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I \in \{0,1\}^E \subset [0,1]^E \subset \mathbb{R}_+^E$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \subseteq [0,1]^E \tag{10.1}$$

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  Examples of P_r^+ are forthcoming.
- Now, take any $x \in P_{\text{ind. set}}$, then we will show that that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this after a few examples of P_r^+.

Vector, modular, incidence

- Recall, any vector $x \in \mathbb{R}^E$ can be seen as a normalized modular function, as for any $A \subseteq E$, we have

$$x(A) = \sum_{a \in A} x_a \tag{10.11}$$

- Given an $A \subseteq E$, define the incidence vector $\mathbf{1}_A \in \{0,1\}^E$ on the unit hypercube as follows:

$$\mathbf{1}_A \stackrel{\text{def}}{=} \left\{ x \in \{0,1\}^E : x_i = 1 \text{ iff } i \in A \right\} \tag{10.12}$$

equivalently,

$$\mathbf{1}_A(j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j \in A \\ 0 & \text{if } j \notin A \end{cases} \tag{10.13}$$

$P_{\text{ind. set}} \subseteq P_r^+$

Lemma 10.3.1 ($P_{\text{ind. set}} \subseteq P_r^+$)

- *If* $x \in P_{\text{ind. set}}$*, then*

$$x = \sum_i \lambda_i \mathbf{1}_{I_i} \tag{10.18}$$

for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

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$$\leq \sum_i \lambda_i \max_{j : I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{10.20}$$

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$$= \max_{j: I_j \subseteq A} \mathbf{1}_{I_j}(E) = \max_{I \in \mathcal{I}} |A \cap I| \tag{10.21}$$

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- *Clearly, for such x, $x \geq 0$.*
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$$= r(A) \tag{10.22}$$
- *Thus, $x \in P_r^+$ and hence $P_{\text{ind. set}} \subseteq P_r^+$.*

Containment: Matroid Independence Polyhedron

- Thus, we have that:

$$
\begin{aligned}
P_{\text{ind. set}} &= \operatorname{conv}\left\{\cup_{I \in \mathcal{I}}\{\mathbf{1}_I\}\right\} \\
&\subseteq P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\}
\end{aligned} \tag{10.23}
$$

Containment: Matroid Independence Polyhedron

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- Therefore, since $\{\mathbf{1}_I : I \in \mathcal{I}\} \subseteq \text{conv}\left\{\bigcup_{I \in \mathcal{I}}\{\mathbf{1}_I\}\right\} = P_{\text{ind. set}} \subseteq P_r^+$, we have that

$$\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{10.24}$$

$$\leq \max\left\{w^\intercal x : x \in P_r^+\right\} \tag{10.25}$$

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$$\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{10.24}$$

$$\leq \max\left\{w^\intercal x : x \in P_r^+\right\} \tag{10.25}$$

- In fact, the two polyhedra $P_{\text{ind. set}}$ and P_r^+ are identical (and thus both are polytopes). We'll show this in the next few theorems.

Maximum weight independent set via greedy weighted rank

Theorem 10.3.2

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{10.26}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{10.27}$$

Maximum weight independent set via weighted rank

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = (w_1 - w_2)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3)\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + (w_{n-1} - w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{10.28}$$

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Maximum weight independent set via weighted rank

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- If we can take w in non-increasing order ($w_1 \geq w_2 \geq \cdots \geq w_n$), then each coefficient of the vectors is non-negative (except possibly the last one, w_n).

...

Maximum weight independent set via weighted rank

Proof.

- Again assuming $w \in \mathbb{R}_+^E$, w.l.o.g. order elements of V non-increasing by w so $(v_1, v_2, \ldots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$

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- Define the sets U_i based on this order as follows, for $i = 0, \dots, n$

$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \dots, v_i\} \tag{10.29}$$

Note that $U_0 = \emptyset$ and

$$\mathbf{1}_{U_0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{1}_{U_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{1}_{U_\ell} = \begin{pmatrix} \left.\begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix}\right\} \ell\times \\ \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} (n-\ell)\times \end{pmatrix}, \text{ etc.}$$

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- Define the set I as those elements where the rank increases, i.e.:

$$I \stackrel{\text{def}}{=} \{v_i | r(U_i) > r(U_{i-1})\}. \tag{10.30}$$

Hence, given an i with $v_i \notin I$, $r(U_i) = r(U_{i-1})$.

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- Therefore, I is the output of the greedy algorithm for $\max\{w(I) | I \in \mathcal{I}\}$. *since items v_i are ordered decreasing by $w(v_i)$, and we only choose the ones that increase the rank, which means they don't violate independence.*

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- Therefore, I is the output of the greedy algorithm for $\max\{w(I) | I \in \mathcal{I}\}$.
- And therefore, I is a maximum weight independent set (can even be a base, actually).

...

Maximum weight independent set via weighted rank

Proof.

- Now, we define λ_i as follows

$$0 \leq \lambda_i \stackrel{\text{def}}{=} w(v_i) - w(v_{i+1}) \text{ for } i = 1, \dots, n-1 \tag{10.31}$$

$$\lambda_n \stackrel{\text{def}}{=} w(v_n) \tag{10.32}$$

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- Since we ordered $v_1, v_2, \dots$ non-increasing by w, for all i, and since $w \in \mathbb{R}_+^E$, we have $\lambda_i \geq 0$

$\square$

Linear Program LP

Consider the linear programming primal problem

$$\begin{aligned} \text{maximize} \quad & w^\intercal x & & \\ \text{subject to} \quad & x_v \geq 0 & & (v \in V) \\ & x(U) \leq r(U) & & (\forall U \subseteq V) \end{aligned} \tag{10.35}$$

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And its convex dual (note $y \in \mathbb{R}_+^{2^n}$, y_U is a scalar element within this exponentially big vector):

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Thanks to strong duality, the solutions to these are equal to each other.

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- This is identical to the problem

$$\max w^\intercal x \text{ such that } x \in P_r^+ \tag{10.38}$$

where, again, $P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\}$.

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where, again, $P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\}$.

- Therefore, since $P_{\text{ind. set}} \subseteq P_r^+$, the above problem can only have a larger solution. I.e.,

$$\max w^\intercal x \text{ s.t. } x \in P_{\text{ind. set}} \leq \max w^\intercal x \text{ s.t. } x \in P_r^+. \tag{10.39}$$

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{10.40}$$

$$\leq \max\{w^\intercal x : x \in P_r^+\} \tag{10.41}$$

$$\stackrel{\text{def}}{=} \alpha_{\text{min}} = \min\left\{\sum_{U \subseteq V} y_U r(U) : \forall U, y_U \geq 0; \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w\right\} \tag{10.42}$$

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- Theorem 10.3.2 states that

$$\max\{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{10.43}$$

for the chain of U_i's and $\lambda_i \geq 0$ that satisfies $w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i}$ (i.e., the r.h.s. of Eq. 10.43 is feasible w.r.t. the dual LP).

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- Therefore, we also have $\max\{w(I) : I \in \mathcal{I}\} \le \alpha_{\min}$ and

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- Hence, we have the following relations:

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- Therefore, all the inequalities above are equalities.
- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r^+ = P_{\text{ind. set}}$

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- Therefore, all the inequalities above are equalities.
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- That is, we have just proven:

Theorem 10.3.3

$$P_r^+ = P_{\textit{ind. set}} \tag{10.45}$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.

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$$P_{\text{ind. set}} = \operatorname{conv}\left\{\cup_{I \in \mathcal{I}}\{\mathbf{1}_I\}\right\} \tag{10.46}$$

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- Now take the rank function r of M, and define the following polytope:

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Theorem 10.3.4

$$P_r^+ = P_{\textit{ind. set}} \tag{10.48}$$

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- In fact, considering equations starting at Eq 10.40, the LP problem with exponential number of constraints $\max\{w^\intercal x : x \in P_r^+\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

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The LP problem $\max \{w^\intercal x : x \in P_r^+\}$ *can be solved exactly using the greedy algorithm.*

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Note that this LP problem has an exponential number of constraints (since P_r^+ is described as the intersection of an exponential number of half spaces).

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- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

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- Note the third requirement, $x(V) = r(V)$.

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- By essentially the same argument as above (Exercise:), we can shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 10.49- 10.51 above.

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- What does this look like? The base polytope.

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Theorem 10.3.6

The spanning set polytope is determined by the following equations:

$$0 \leq x_e \leq 1 \qquad \text{for } e \in E \tag{10.52}$$

$$x(A) \geq r(E) - r(E \setminus A) \qquad \text{for } A \subseteq E \tag{10.53}$$

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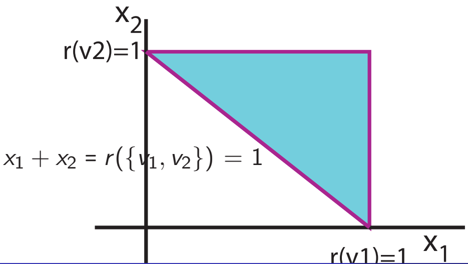
- Example of spanning set polytope in 2D.



Spanning set polytope

Proof.

- Recall that any A is spanning in M iff $E \setminus A$ is independent in M^* (the dual matroid).

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- Recall that any A is spanning in M iff $E \setminus A$ is independent in M^* (the dual matroid).
- For any $x \in \mathbb{R}^E$, we have that

$$x \in P_{\text{spanning}}(M) \Leftrightarrow 1 - x \in P_{\text{ind. set}}(M^*) \tag{10.54}$$

as we show next ...

...

Spanning set polytope

. . . proof continued.

- This follows since if $x \in P_{\text{spanning}}(M)$, we can represent x as a convex combination:

$$x = \sum_i \lambda_i \mathbf{1}_{A_i} \tag{10.55}$$

where A_i is spanning in M.

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- Consider

$$\mathbf{1} - x = \mathbf{1}_E - x = \mathbf{1}_E - \sum_i \lambda_i \mathbf{1}_{A_i} = \sum_i \lambda_i \mathbf{1}_{E \setminus A_i}, \tag{10.56}$$

which follows since $\sum_i \lambda_i \mathbf{1} = \mathbf{1}_E$, so $\mathbf{1} - x$ is a convex combination of independent sets in M^* and so $\mathbf{1} - x \in P_{\text{ind. set}}(M^*)$.

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Spanning set polytope

...proof continued.

- which means, from the definition of $P_{\text{ind. set}}(M^*)$, that

$$\mathbf{1} - x \geq 0 \tag{10.57}$$

$$\mathbf{1}_A - x(A) = |A| - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E \tag{10.58}$$

And we know the dual rank function is

$$r_{M^*}(A) = |A| + r_M(E \setminus A) - r_M(E) \tag{10.59}$$

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□

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- By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it it worth showing a result for a general submodular function first.
- Henceforth, we will skip between submodular functions and matroids, each lecture talking less about matroids specifically and taking more about submodular functions more generally ...

Maximal points in a set

- Regarding sets, a subset X of S is a maximal subset of S possessing a given property $\mathfrak{P}$ if X possesses property $\mathfrak{P}$ and no set properly containing X (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\mathfrak{P}$.

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- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector x is maximal within P if it is the case that for any $\epsilon > 0$, and for all directions $e \in E$, we have that

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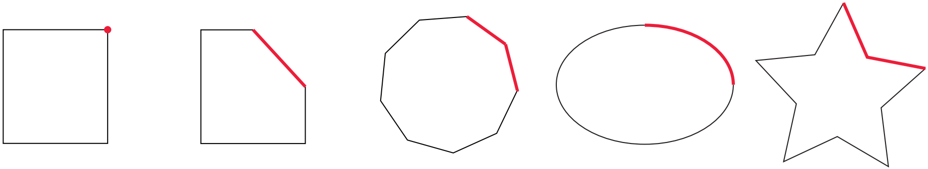
- Examples of maximal regions (in red)



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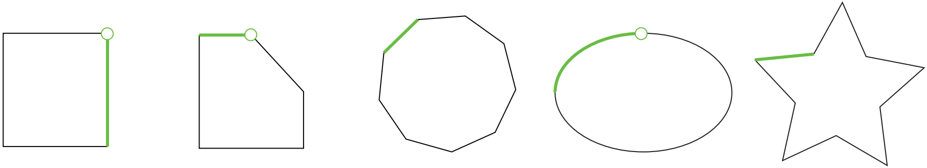

Review from Lecture 6

- The next slide comes from Lecture 6.

Matroids, independent sets, and bases

- Independent sets: Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called independent if $A \in \mathcal{I}$ and otherwise A is called dependent.
- A base of $U \subseteq E$: For $U \subseteq E$, a subset $B \subseteq U$ is called a base of U if B is inclusionwise maximally independent subset of U. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.
- A base of a matroid: If $U = E$, then a "base of E" is just called a base of the matroid M (this corresponds to a basis in a linear space, or a spanning forest in a graph, or a spanning tree in a connected graph).

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 10.4.1 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

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Here, by y being "maximal", we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P, and a subvector of x).

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In still other words: y is a P-basis of x if:

1. $y \leq x$ (y is a subvector of x); and
2. $y \in P$ and $y + \epsilon \mathbf{1}_e \notin P$ for all $e \in E$ where $y(e) < x(e)$ and $\forall \epsilon > 0$ (y is maximal P-contained).

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max\left\{|I| : I \subseteq A, I \in \mathcal{I}\right\} = \max_{I \in \mathcal{I}} |A \cap I| \tag{10.62}$$

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where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$), and where $(x \wedge y) \in \mathbb{R}_+^E$ has $(x \wedge y)(i) = \min(x(i), y(i))$.

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- In general, might be hard to compute and/or have ill-defined properties. Next, we look at an object that restrains and cultivates this form of rank.

Polymatroidal polyhedron (or a "polymatroid")

Definition 10.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
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- Condition 3 restated: That is for any two distinct maximal vectors $y^1, y^2 \in P$, with $y^1 \leq x$ & $y^2 \leq x$, with $y^1 \neq y^2$, we must have $y^1(E) = y^2(E)$.

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- Condition 3 restated (again): For every vector $x \in \mathbb{R}_+^E$, every maximal independent (i.e., $\in P$) subvector y of x has the same component sum $y(E) = \text{rank}(x)$.

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- Condition 3 restated (yet again): All P-bases of x have the same component sum.

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- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Matroid and Polymatroid: side-by-side

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3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.
4. any maximal set I in $\mathcal{I}$, bounded by another set A, has the same matroid rank (any maximal independent subset $I \subseteq A$ has same size $|I|$).

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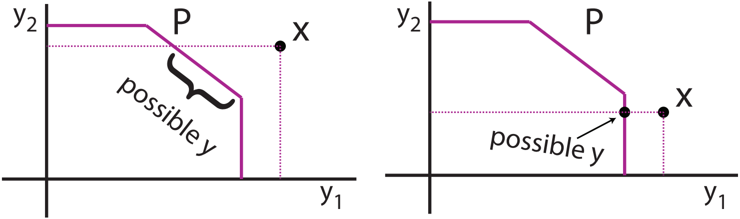
4. any maximal vector y in P, bounded by another vector x, has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).

Polymatroidal polyhedron (or a "polymatroid")



Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 =$ const.

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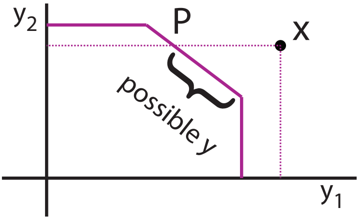

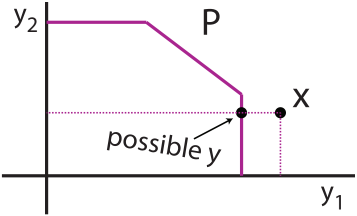

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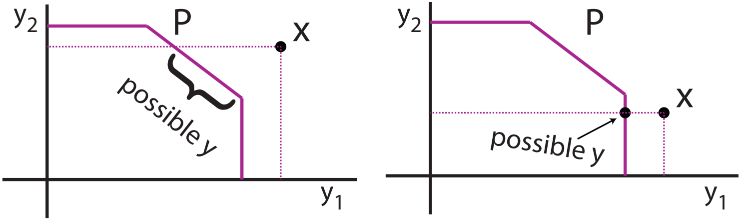
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such y must have the same value $y(E)$.

Polymatroidal polyhedron (or a "polymatroid")



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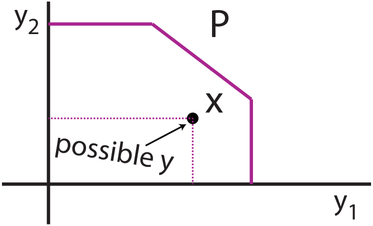
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such y must have the same value $y(E)$.
- On the right, there is only one maximal $y \in P$. Since there is only one, the condition on the same value of $y(E), \forall y$ is vacuous.

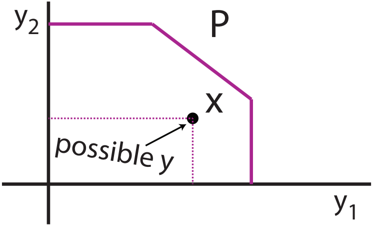
Polymatroidal polyhedron (or a “polymatroid”)



$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then x is its own P-basis, i.e., it is a self P-basis.

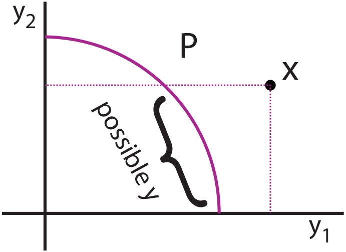
Polymatroidal polyhedron (or a "polymatroid")

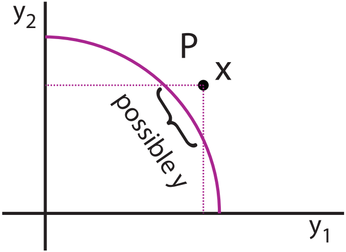


$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then x is its own P-basis, i.e., it is a self P-basis.
- In a matroid, a base of A is the maximally contained independent set. If A is already independent, then A is a self-base of A (as we saw in previous Lectures)

Polymatroid as well?

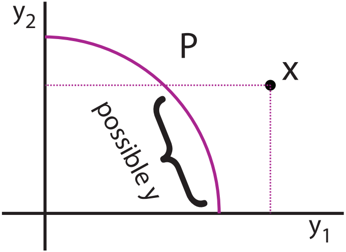


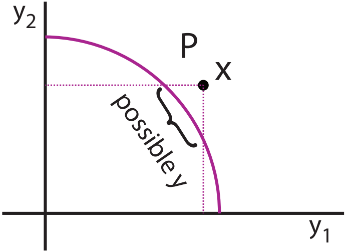


Left and right: $\exists$ multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such y must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 =$ const. $\neq y_1 + y_2$, we see this is not a polymatroid.

Polymatroid as well? no





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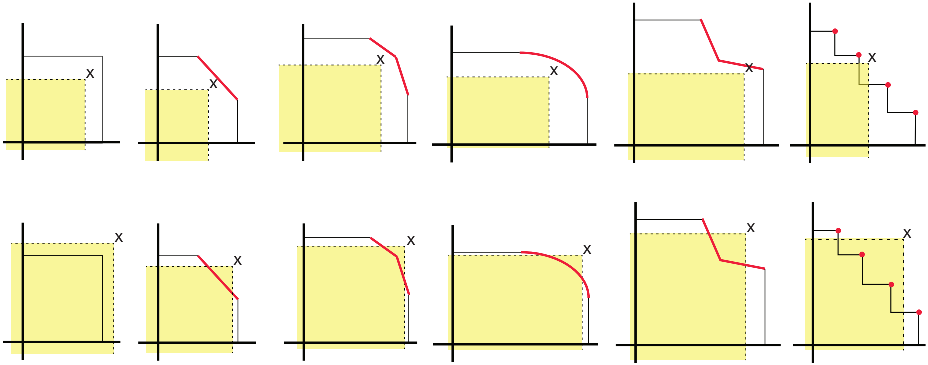
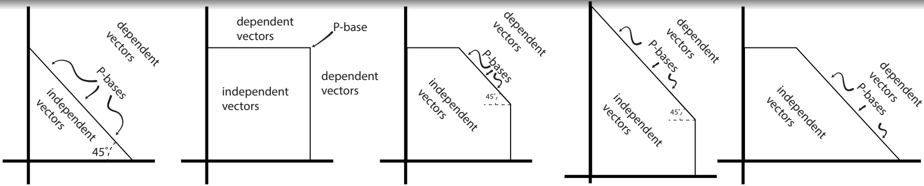
- On the right, we have a similar situation, just the set of potential values that must have the $y(E)$ condition changes, but the values of course are still not constant.

Other examples: Polymatroid or not?



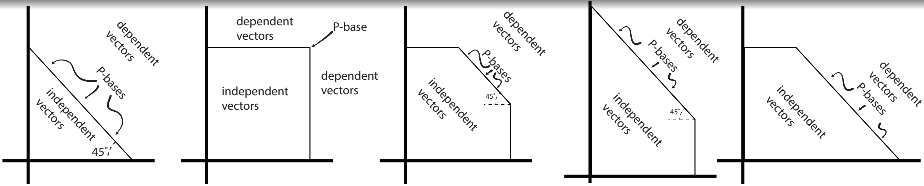
Some possible polymatroid forms in 2D



It appears that we have five possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2

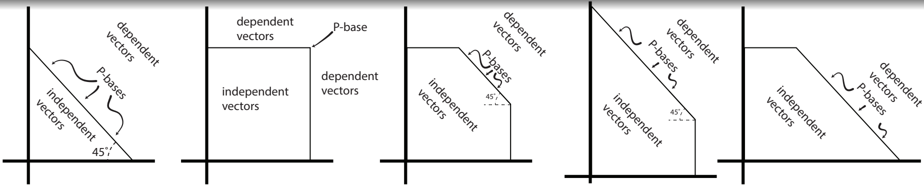
Some possible polymatroid forms in 2D



It appears that we have five possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2
2. Next: full independence between v_1 and v_2

Some possible polymatroid forms in 2D



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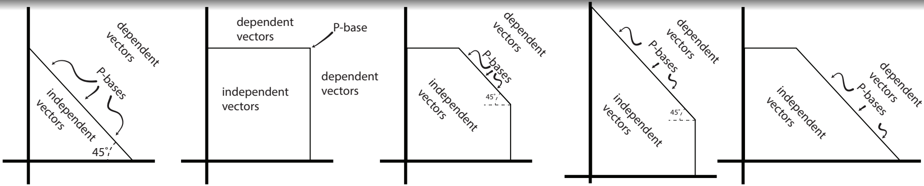
2. Next: full independence between v_1 and v_2
3. Next: partial independence between v_1 and v_2

Some possible polymatroid forms in 2D



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4. Right two: other forms of partial independence between v_1 and v_2

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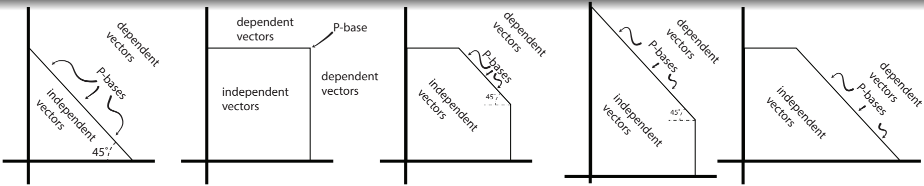

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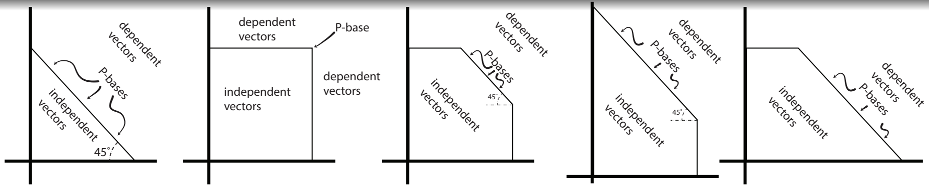

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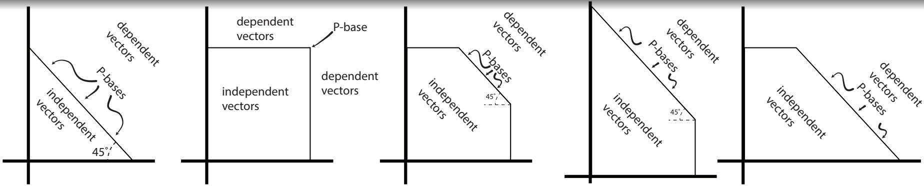

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- The P-bases (or single P-base in the middle case) are as indicated.
- Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.
- The set of P-bases for a polytope is called the base polytope.

Polymatroidal polyhedron (or a "polymatroid")

- Note that if x contains any zeros (i.e., suppose that $x \in \mathbb{R}_+^E$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so S indicates the non-zero elements, or $S = \text{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$.

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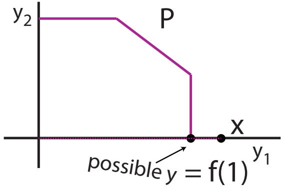
- Therefore, in this case, it is the non-zero elements of x, corresponding to elements S (i.e., the support $\text{supp}(x)$ of x), determine the common component sum.
- For the case of either $x \notin P$ or right at the boundary of P, we might give a "name" to this component sum, lets say $f(S)$ for any given set S of non-zero elements of x. We could name $\text{rank}(\frac{1}{\epsilon}\mathbf{1}_S) \triangleq f(S)$ for ϵ small enough. What kind of function might f be?



Polymatroid function and its polyhedron.

Definition 10.4.4

A polymatroid function is a real-valued function f defined on subsets of E which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron P_f^+ associated with a polymatroid function as follows

$$P_f^+ = \left\{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\right\} \tag{10.64}$$

$$= \left\{y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E\right\} \tag{10.65}$$

Associated polyhedron with a polymatroid function

$$P_f^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\right\} \tag{10.66}$$

- Consider this in three dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \tag{10.67}$$
$$x_1 \leq f(\{v_1\}) \tag{10.68}$$
$$x_2 \leq f(\{v_2\}) \tag{10.69}$$
$$x_3 \leq f(\{v_3\}) \tag{10.70}$$
$$x_1 + x_2 \leq f(\{v_1, v_2\}) \tag{10.71}$$
$$x_2 + x_3 \leq f(\{v_2, v_3\}) \tag{10.72}$$
$$x_1 + x_3 \leq f(\{v_1, v_3\}) \tag{10.73}$$
$$x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\}) \tag{10.74}$$

Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.

Associated polyhedron with a polymatroid function

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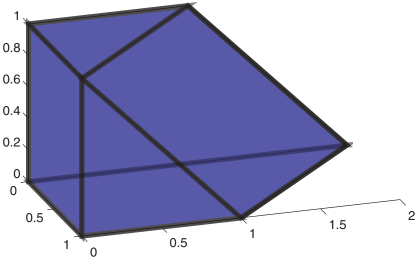
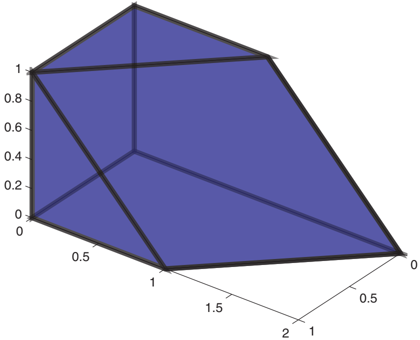
- Observe: P_f^+ (at two views):





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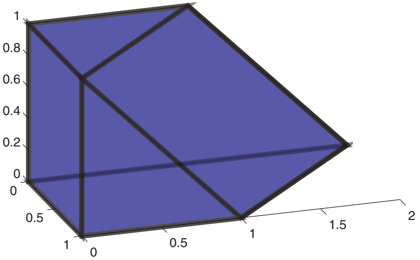

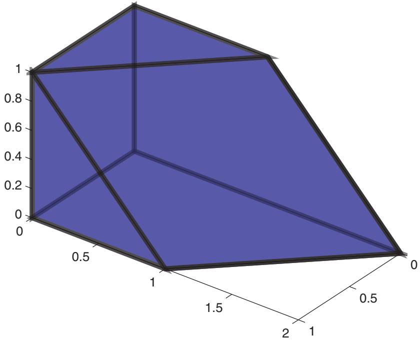

- which axis is which?

Associated polyhedron with a polymatroid function

- Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.

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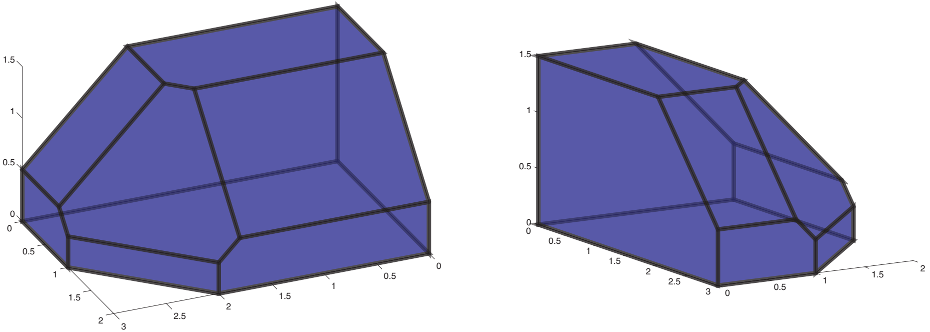

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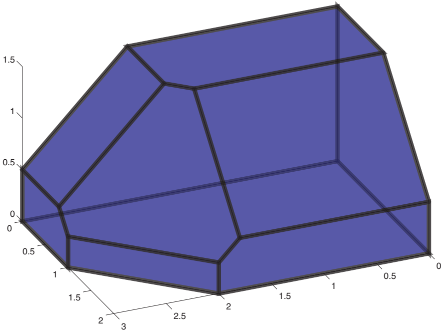

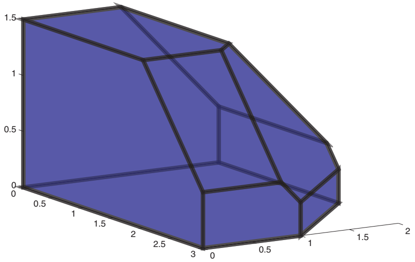

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Associated polyhedron with a polymatroid function

- Consider modular function $w : V \to \mathbb{R}_+$ as $w = (1, 1.5, 2)^\intercal$, and then the submodular function $f(S) = \sqrt{w(S)}$.

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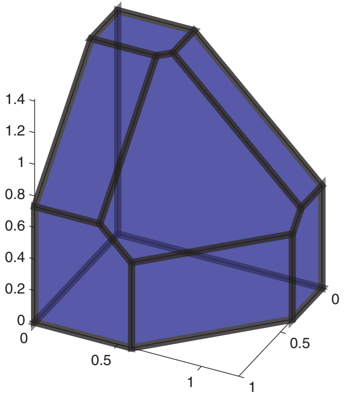

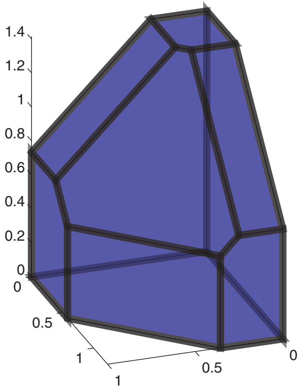

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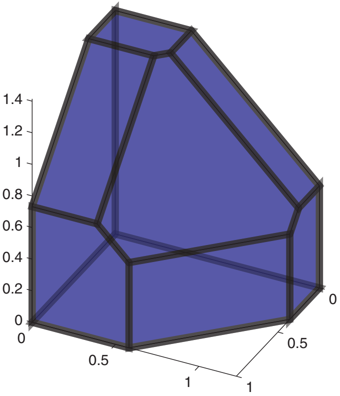

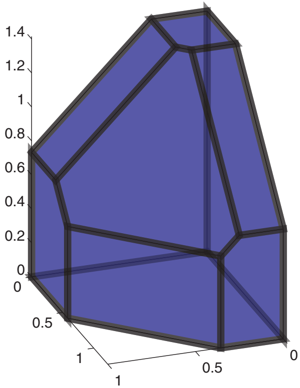

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Associated polytope with a non-submodular function

- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$.

Associated polytope with a non-submodular function

- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular?

Associated polytope with a non-submodular function

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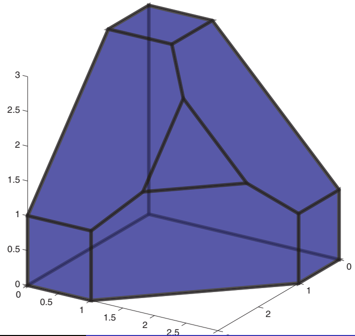
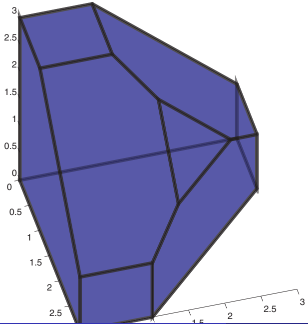
- Observe: P_f^+ (at two views), maximal independent subvectors not constant rank, hence not a polymatroid.





A polymatroid vs. a polymatroid function's polyhedron

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- Is there any relationship between these two polytopes?
- In the next theorem, we show that any P_f^+-basis has the same component sum, when f is a polymatroid function, and P_f^+ satisfies the other properties so that P_f^+ is a polymatroid.

A polymatroid function's polyhedron is a polymatroid.

Theorem 10.5.1

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \textit{rank}(x) \triangleq \max\left(y(E) : y \le x, y \in P_f^+\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{10.76}$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

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Taking $E \setminus B = \operatorname{supp}(x)$ (so elements B are all zeros in x), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank}\left(\frac{1}{\epsilon}\mathbf{1}_{E\setminus B}\right) = f(E \setminus B) = \max\left\{y(E \setminus B) : y \in P_f^+\right\} \tag{10.77}$$

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In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

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Proof.

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- Doing so will thus establish that P_f^+ is a polymatroid.

...

A polymatroid function's polyhedron is a polymatroid.

... proof continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) \leq f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case,

$$\min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{10.78}$$
$$= x(E) + \min\left(f(E \setminus A) - x(E \setminus A) : A \subseteq E\right) \tag{10.79}$$
$$= x(E) + \min\left(f(A) - x(A) : A \subseteq E\right) \tag{10.80}$$
$$= x(E) \tag{10.81}$$

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- When $x \in P_f^+$, $y = x$ is clearly the solution to $\max\left(y(E) : y \leq x, y \in P_f^+\right)$, so this is tight, and $\text{rank}(x) = x(E)$.
- This is a value dependent only on x, a self basis, unique P_f^+-base.

...

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- 2nd trivial case: $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P_f^+$ every direction),

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- Then for any order $(a_1, a_2, \dots)$ of the elements and $A_i \triangleq (a_1, a_2, \dots, a_i)$, we have $x(a_i) \geq f(a_i) \geq f(a_i | A_{i-1})$, the second inequality by submodularity.

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$$\min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{10.82}$$

$$= x(E) + \min\left(f(A) - x(A) : A \subseteq E\right) \tag{10.83}$$

$$= x(E) + \min\left(\sum_i f(a_i | A_{i-1}) - \sum_i x(a_i) : A \subseteq E\right) \tag{10.84}$$

$$= x(E) + \min\left(\sum_i \underbrace{\Big(f(a_i | A_{i-1}) - x(a_i)\Big)}_{\leq 0} : A \subseteq E\right) \tag{10.85}$$

$$= x(E) + f(E) - x(E) = f(E) = \max(y(E) : y \in P_f^+). \tag{10.86}$$

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- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.

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$$y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{10.87}$$

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- For any P_f^+-basis y^x of x, and any $A \subseteq E$, we have weak relationship:

$$\begin{aligned} y^x(E) &= y^x(A) + y^x(E \setminus A) && (10.88) \\ &\leq x(A) + f(E \setminus A). && (10.89) \end{aligned}$$

This follows since $y^x \leq x$ *and since* $y^x \in P_f^+$.

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- This ensures

$$\max\left(y(E) : y \leq x, y \in P_f^+\right) \leq \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{10.90}$$

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- Given an A where equality in Eqn. (10.89) holds, above min result follows. ...

A polymatroid function's polyhedron is a polymatroid.

...proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C)$$

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which requires equality everywhere above.

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- Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.

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  which requires equality everywhere above.
- Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.
- For $y \in P_f^+$, it will be ultimately useful to define this lattice family of tight sets: $\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\}$.

. . .

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- Also, we define $\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$, so $y(\text{sat}(y)) = f(\text{sat}(y))$.

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- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$ (since if $e \in T \in \mathcal{D}(y^x)$, then $e \in \text{sat}(y^x)$).

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- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).

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- Hence, we have

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- Hence, we have

$$y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \tag{10.95}$$

- So we identified the A to be the elements that are non-tight, and achieved the min, as desired. $\square$

A polymatroid is a polymatroid function's polytope

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Theorem 10.5.2

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\mathrm{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{10.96}$$

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We have already proven this as part of Theorem 10.5.1 □

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Also recall the definition of $\mathrm{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{10.97}$$

Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{10.98}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{10.99}$$

Hence,

$$x \vee y \triangleq \left(\max\Big(x(e_1), y(e_1)\Big), \max\Big(x(e_2), y(e_2)\Big), \dots, \max\Big(x(e_n), y(e_n)\Big) \right)$$

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- From this, we can define things like an lattices, and other constructs.